ATOM

AUTOBIOGRAPHICAL NOTES

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Editorial Note—These notes are excerpts from a manuscript composed by Professor Bliss early in 1938. His purpose in setting them down was principally to save trouble for his biographers. The parts printed here include some remarks on mathematics and mathematical education, mathematicians and universities, which seem likely to be interesting to readers of the MONTHLY. A biographical note, with a bibliography and comments on Bliss's scientific work, by L. M. Graves, has appeared in the Bulletin of the American Mathematical Society, vol. 58, 1952, pp. 251–264.

Bliss was born in Chicago and attended the Chicago public schools. His interests were quite general until his second year at the University of Chicago when he studied the calculus; his serious interest in mathematics began at this time and he decided to make it his principal subject.

1. Graduate work at Chicago. In the graduate school at the University of Chicago my work at first was mostly in mathematical astronomy under F. R. Moulton. It was fascinating to me, and I owe to Professor Moulton a very great debt for his inspiring teaching, and especially for his encouragement in the writing of my first published paper. It was entitled "The motion of a heavenly body in a resisting medium" and had an influence on myself quite out of proportion to the value of the paper itself. At the end of my first graduate year I applied for a fellowship in astronomy which was not granted, though I understood that Professor Moulton recommended it. My failure to secure this appointment was a great disappointment to me, in view of my somewhat difficult financial circumstances, but the effect on me was undoubtedly good because it caused me to take stock of my own interests and plans for the future. After some careful consideration I concluded that it was really the mathematics which had the great attraction for me, and I decided to try for a Ph.D. in that subject rather than in astronomy. I have never regretted the decision, though it seems clear to me that one could find great satisfaction in any domain of applied mathematics. I must have decided to make mathematics my principal subject before the end of my first graduate year, because I received the master's degree in mathematics, apparently in the summer quarter of 1898.

The three leading men in mathematics at the University of Chicago at that time (1898) were E. H. Moore, O. Bolza, and H. Maschke. They supplemented each other beautifully. Moore was the fiery enthusiast, keenly interested in the popular mathematical research movements of his day, one after another, at that time especially in group and field theory. Bolza was a product of the meticulous German school of analysis led by Weierstrass. He was an able, aggressive, and widely read research scholar. Maschke was a geometer, much more easy-going than the other two, but brilliant in his research, and the best lecturer on geometry to whom I have ever listened. Under the leadership of these three men Chicago was unsurpassed at that time in America as an institution for the study of higher mathematics. With such representatives of the best mathematical scholarship near at hand one did not need to go abroad to study. Like many other members of our present American mathematical community I am greatly indebted to Moore and Maschke and Bolza for the example of their distinguished

research and scholarship and their continuously friendly and encouraging interest.

My last two years of graduate study passed rapidly, full of the inspiration and hard work well known to every one who has been a candidate for the doctor's degree. Bolza had a beautifully written record, made by himself, of the now famous course in the calculus of variations which Weierstrass had given in 1879, and let me make a copy of it for my own use. It is a course which contains most important results, quite new in 1879, and which has had an unusually wide influence, though in 1898 it was relatively little known. A record of it over Weierstrass' own name was first published in 1927 in volume VII of his Mathematische Werke. The possession of this masterpiece, and Bolza's fascinating lectures, gave me a dominant interest in the calculus of variations, though I was exposed to many other mathematical theories and thoroughly enjoyed them. I wrote my doctor's thesis under Bolza's direction on "The geodesic lines on the anchor ring." It was a study of the forms of these curves on the anchor ring and a classification of them according to their possession or non-possession of conjugate points. I have found the paper a useful one and from my correspondence concerning it at various times believe that others have had the same experience.

2. Early teaching experience. My first teaching position was in 1900 at the University of Minnesota. I enjoyed the students and the teaching from the start. Kneser's *Vorlesungen über Variationsrechnung* had just appeared. It was the only treatise on the subject which contained an account of the ideas of Weierstrass, and was not any too complete in that respect either. I determined to make myself as far as possible an expert in the theory of the calculus of variations, and began by reading Kneser. It was the kind of a program which I still recommend strongly to my young graduate students. If one expects to do mathematical research he must first be a master of the literature of some field, or some portion of a field. When he has become such a master he will find plenty of research questions pressing upon him for answer. Many students never reach this stage, either because they are early distracted by other interests, or because they are too anxious to do research and try to do it without having the fundamental scholarship which justifies it. In my case I found the next two years of absorbing scientific interest. I read Kneser and much other literature of the calculus of variations, and wrote my first completely independent research paper "The second variation when one end-point is variable." It contained an analysis of the movement of a focal point of a curve transversal to an extremal when the curvature of the transversal curve varies. I believe that the first paper a student writes after he has received his Ph.D., or at any time as a result entirely of his own interest and enthusiasm, is the most important one for him. In my case the paper just mentioned above was the one which first gave me some confidence that I might in time become a productive scholar.

3. Post-graduate work abroad. After two years at Minnesota the future there did not look very bright for me, though the years themselves had been

most interesting and profitable ones. My father had died in 1900, but I had saved some money out of my very moderate salary, and my mother offered to supplement this out of her own slender resources in order that I might have the benefit of a year of study abroad. It was something which she never should have done, and which can never be repaid except by helping to provide another such opportunity for another such student. I spent all of my year in Göttingen except for brief visits to Berlin, Italy, Paris, and England. The leaders in Göttingen at that time were Klein, Hilbert, and Minkowski, a notable group. Among the younger men were Zermelo, E. Schmidt, Abraham, Fejer, and Caratheodory, certainly inspiring comrades. As a visitor with his Ph.D. already behind him I was invited to attend the Mathematische Gesellschaft which met every other week. Klein was the dominant figure in the Gesellschaft, a marvel of wide acquaintance with mathematics and mathematicians, and ready with many interesting comments on their books and personalities. After the meeting was over Hilbert and Minkowski would lead a party of a dozen or so to a neighboring cafe where the evening would be spent in lively mathematical discussions.

My year at Göttingen was a most profitable one. I listened to lecture courses by Hilbert, Minkowski, and Klein, and took an active part as a student in the seminar of Hilbert and Minkowski. Hilbert was lecturing on Mechanik der Kontinua, Minkowski on Minimalflächen, and Klein on his Encyklopädie. My reports in the seminar were on some problems in conformal representation in which Hilbert was interested. None of these was closely associated with my chief interest, the calculus of variations, but I found in the excellent library of the mathematical reading room most interesting records of courses in the calculus of variations by Weierstrass, Hilbert, Sommerfeldt, and Zermelo. These alone would have justified my trip to Europe. The profit becomes much greater if one adds to them the inspiration of distinguished professors, the benefit of acquaintance with the group of younger men many of whom afterward became famous, and the great pleasure of lifelong friendship with Max Mason and O. D. Kellogg, then young American candidates for the doctor's degree at Göttingen. A paper which emerged from my studies of that year was a discussion of necessary and sufficient conditions for a minimum for a problem of the calculus of variations in the plane with both end-points variable.

In their seminar Hilbert and Minkowski first assigned reports to students in pairs. My colleague was an assistant in a neighboring department of the University who had a quite remarkable acquaintance with mathematical books. I was much discouraged at my own ignorance after our first conference, but I soon found that he had very moderate command of mathematical techniques. I have thought sometimes that our relatively informal exercises and reports in our graduate courses in America tend to develop control of technique earlier than the more formal lecture courses of the German universities. At any rate after our first report I was called upon to make all of the rest of them. It was rather rough going because of my poor German. One day Hilbert interrupted

me suddenly with the remark "Ach Herr Bliss, dies Mal haben sie sich gierrt," whereupon both he and Minkowski rushed to the board, as was their custom, and began writing and talking excitedly with no regard for each other, Hilbert with his face very close to the board because he was so nearsighted, all to the great delight of the students. Presently Hilbert turned to me, as much pleased as I was to find that there had been no mistake, and said that I must have meant "eine eingeschnittene Ellipse." An ellipse must be cut in from the ends of the major axes to the foci in order to be represented conformally on a rectangle by the method I was trying to explain. The word "eingeschnitten" was not in my vocabulary at that time, but I venture to say that it is now permanently a part of it.

Klein was a dominant figure in German mathematics, and indeed in the German educational world in general. As a younger man he did beautiful research in many domains of pure mathematics. When he grew older he turned almost entirely to applied mathematics, in which his research has seemed to me much less significant, and became rather critical, so it seemed to me, of research in pure mathematics. When I was in Göttingen he looked really ill, and though he lived many years after that I think that he had always to be very careful of his health. One day I was asked to give a paper at the Gesellschaft and presented the paper mentioned above. When I had finished Klein rather took my breath away by asking "But, Herr Bliss, who would be interested in this?" That was for the moment a poser to me, but I tried to explain my belief that one who had been especially a student of the calculus of variations might find something of interest in the results of the paper. It seems that I mixed the words "man" and "sie" so that my remark had rather the meaning that "If you knew anything about the calculus of variations you would be interested." I still remember the breathless moment which followed my remark, incomprehensible to me at that time but comprehensible enough several years later when Max Abraham retold the incident to some of us with, I hope, much exaggeration. Certainly I could never intentionally be disrespectful to one who has wielded the uniquely distinguished influence of Klein. As for my paper I must say in self defense that I have had many inquiries for the reprints which unfortunately were lost somewhere between Germany and the United States. As I look at the paper again from time to time it seems to me a rather good one, the first relatively complete treatment of a problem of the calculus of variations with two variable end-points.

4. Later teaching positions. In the autumn of 1903 a position at Minnesota would still have been open to me, but Professor E. H. Moore, head of the mathematical department of the University of Chicago, wrote to me that I could have an associateship there for one year if I was interested. It seemed to me an excellent opportunity to spend another year in an inspiring mathematical environment, and I accepted his suggestion with much alacrity. My anticipation proved to be well justified. My sojourn abroad had not weakened, but had strengthened, my respect for the Chicago department of mathematics, and for my

professors there, and I enjoyed and profited much from my associations with them as a member of the faculty during the year 1903–4.

In the autumn of 1904 I went as assistant professor to the University of Missouri where E. R. Hedrick was head of the department of mathematics. It was my first experience with life in a small town and I enjoyed my surroundings immensely. Hedrick was ideally congenial as a leader. While I was there he organized an association of Missouri teachers of mathematics. The first meeting was held at the University in Columbia, Missouri, and we all helped as much as we could to make it a success. Hedrick had, however, a knack for making things go irrespective of our assistance. I had hardly become a part of the University circle in Columbia when in the spring of 1905 I had an invitation to go to Princeton. It was with great regret that I left, carrying with me very permanent memories of a most happy year and of my friendly associations with Hedrick. In one of my classes W. A. Hurwitz, now professor at Cornell University, was a student. He was extraordinarily precocious and able.

In the autumn of 1905 the then new preceptorial system at Princeton was greatly extended. About fifty new "preceptors" were added to the faculty. I was one of them. Some of us did not like the word "preceptor" very much, because it suggested tutor or assistant and had no established meaning in the educational world. We were assured that we were the equals of assistant professors elsewhere and that we might think of ourselves as such. I always did so and have always listed myself as such to a world which has now pretty much forgotten the preceptorial system. Princeton is an exceedingly friendly but somewhat conservative place, and I have doubts if the fifty preceptors really became an integral part of the community during the three years that I was there. Those of us in mathematics had a most interesting time, however, with our science and our students.

H. B. Fine was Dean and also head of the department of mathematics during my sojourn in Princeton. He was much beloved in his community, not aggressive in mathematical research, but most sympathetic and encouraging to the younger men in his department. We regarded him with great affection. Frequently on Saturday nights the younger men in the department met in the rooms of one or another of them for refreshments and mathematical discussions. The discussions were the liveliest and in many way the most interesting ones in which I have ever taken part. Eisenhart, Veblen, J. W. Young and R. L. Moore were among the leaders and my friendship for them and admiration of their scholarship have remained with me permanently. During one of my years at Princeton Jeans was a visiting professor. He was about our own age but very unapproachable. We tried to give him a good hearing in his lectures on Dynamical Theory of Gases, but anything like free discussion proved to be impossible. I learned more of him later by reading his book on electro-magnetic theory, but have never been enthusiastic about his writing as viewed from the standpoint of pure as contrasted to applied mathematics.

5. Teaching at Chicago. About the time I was in Princeton I spent most of my summers teaching in universities other than Princeton. In 1907 a summer term at Wisconsin, where I renewed my associations with Mason and his family, gave me much pleasure. He was at Yale while I was at Princeton and I saw him there from time to time also. But most of my summer teaching was in the Department of Mathematics at Chicago. In the early part of 1908 Maschke died, and during the summer of that year, while teaching at Chicago, I was offered an associate professorship there. Maschke's loss to his Department was a very great one, in fact irreparable, and I was sure that they would have preferred an older man to replace him. But the challenge to me was one which I could not refuse and I was anxious to accept. Professor Fine at Princeton was most generous in his willingness to facilitate my move, and after teaching some weeks at Princeton in the autumn of 1908, while he made readjustments in his staff and schedule, I joined the Department at Chicago and have remained there ever since.

The next two years were rather strenuous ones for me. I undertook to give some of the advanced courses in geometry which Maschke had always given before, metric differential geometry, analytic projective geometry, higher plane curves. It was most interesting work and I have always appreciated the wider experience in mathematics which this teaching gave me though it took much time. In 1909 Kasner and I were the lecturers at the Princeton Colloquium of the American Mathematical Society. My part of the colloquium was on fundamental existence theorems for implicit functions and ordinary and partial differential equations in the theory of functions of real variables. I have always hoped to find time to pursue much further the theory of singular points of real transformations of the plane which I developed in those lectures.

In 1910 Bolza withdrew from the department of mathematics at Chicago and went back to Germany to live in Freiburg in Baden. His mother there was quite old and was not expected to live long, though as a matter of fact she did live for many years. Bolza's departure was another great loss to Chicago. Wilczynski, then of Illinois, had made a name for himself in projective differential geometry, and we were fortunate in securing him in the autumn of 1910 as a member of our mathematical staff. His arrival and Bolza's departure brought my teaching back again into my chosen field of analysis.

In 1913 I was made a full professor at the University of Chicago. Slaught as the leader, with Dickson, Moore, and of course many others, were engaged about that time in strengthening the AMERICAN MATHEMATICAL MONTHLY and in the organization of the Mathematical Association of America. In the summer of 1918, after the United States had been drawn into the war, we undertook in our Department to give courses in navigation to men who had been accepted and were waiting in Chicago to enter the naval school there. We had a number of sections. Mine had about one hundred students in it. To my surprise when the course was over the men in it presented me with a very fine watch. I have never felt that acceptance of it was appropriate, since they were the ones who were

making the sacrifices and preparing themselves for service. But I must say that I have valued their gift much, especially as a reminder of their seriously attentive and successful interest.

6. Aberdeen Proving Ground. In late summer of 1918 my old friend Oswald Veblen began writing me from Aberdeen Proving Ground saying that he needed mathematicians. He was major in charge of the Range Firing Section there, with the duty of supervising the preparation of range tables. Our work at Chicago with the prospective naval students was by that time well organized, and it seemed to me the most effective work which I could then do. But I had confidence in Veblen's judgment and after repeated letters decided to join him at Aberdeen as what they called a "scientific expert." Veblen was right, of course, about the need of mathematics in ballistics. The old Siacci theory broke down early in the war because it contained an approximation not justifiable for gun elevations greater than twenty degrees. F. R. Moulton, Veblen, and their associates had devised effective methods of successive approximation for solving the differential equations of trajectories in all cases, but the theory and application of the so-called "differential corrections" of trajectories was still very imperfect. These are corrections made to the normal range to account for the effects of wind, variations from normal in the density of the air and the weights of the projectile and powder charge, and in the case of long ranges for the rotation of the earth. This was my particular field of study, and I was astonished to find that the most advanced mathematics which I knew was needed and effective. The wind correction, for example, is a function of the wind curve when wind velocity is plotted as a function of altitude. It is what the mathematicians sometimes call a functional or a function of a line. The calculation of the differentials of such functionals is a principal problem of my special field, the calculus of variations. I found that the methods of that calculus were effective, and that in many cases with suitable adjustments they reduced by three-fourths the time necessary for the computation of range corrections. The saving for anti-aircraft trajectories was not so great, but the method is applicable there also, and I have left a detailed description of it for that case in a blue print in the archives at the Proving Ground. These results and savings were important for the Range Firing Section because of the large number of computers engaged in the construction of range tables and the constant flow of data into the Section concerning new experimental firings and new types of guns. I am happy to say that there are many applications other than in ballistics of the method which I devised. Some of them I have pursued to my own satisfaction.

7. Post-war years. In 1921 I was elected president of the American Mathematical Society. It is a position of great honor, and also of great responsibility. The Society was having financial difficulties, a condition which seems to be normally associated with healthy progress. E. R. Hedrick suggested a campaign to increase membership. I must confess that I doubted the possibility of success sufficient to be profitable, and that I undertook the labor of the campaign

with great reluctance because of the time it would take from my scientific research interests. Once started, however, both Hedrick and I devoted ourselves to it heartily. The results well justified our efforts, especially as this first campaign of the Society encouraged later and more successful ones which have developed into a continuous membership campaign under the guidance of our skillful secretary, R. G. D. Richardson.

At Chicago, as at other universities, the years following the war were exceedingly crowded ones. There seemed no limit to the number of students who were seeking college educations and higher degrees. Our situation was aggravated by the serious illness and consequent retirement of Wilczynski from active work in 1923. Fortunately we were able to secure E. P. Lane, his very able successor in the field of projective differential geometry, to carry on his work. Our climax was reached in 1927. Advanced graduate courses in one summer quarter sometimes attracted over eighty students, and it seemed almost impossible to take care of the large number of candidates seeking higher degrees in mathematics. Fortunately the head of our Department, E. H. Moore, was conservative about increasing our permanently appointed personnel, for after 1927 there was at first a gradual reduction in the demands upon us, and then a rapid decrease during the depression years. We were fortunate in not having expanded our Department unduly, and in being able to retain our regularly appointed staff intact without salary reductions.

In 1927 Professor Moore had passed the normal retiring age and was anxious to lay aside his administrative responsibilities as head of the Department. The Trustees appointed me chairman, at first without public announcement, and Mr. Moore was retained as nominal head.

In 1927 the University established a number of special professorships, one of which is the Eliakim Hastings Moore Distinguished Service Professorship. L. E. Dickson is the first incumbent. He was one of Moore's earliest students, and his career in mathematical research has been remarkable. I can think of no more appropriate appointment. One of the purposes of our Department has been to provide Dickson with a most favorable environment for the pursuit of his teaching and research interests, and I believe that we have succeeded. In 1933 I was appointed to a similar professorship bearing the name of Martin A. Ryerson. The appointment was a surprise to me and gave me one of the greatest of the pleasurable thrills of my life.

8. Editorial and committee work. In the preceding paragraphs I have not mentioned my editorial work. I was associate editor of the Annals of Mathematics from 1906 to 1908, and of the Transactions of the American Mathematical Society from 1909 to 1916. E. H. Moore, T. S. Fiske, and E. W. Brown, the first chief editors of the Transactions, actually taught many members of our then youthful research community to write in good mathematical style. Others of us who assisted in those earlier days attempted to do the same. I spent days at a time trying to master unfamiliar fields and preparing sometimes

unwelcome advice to authors. The result was that I developed a very great distaste for editorial work. The papers which especially irked me were theses from other universities than my own which I thought should have been more carefully edited at home. In 1916 a chief editorship of the Transactions was offered to me and I declined it. My decision was probably a mistake, since many of our best mathematicians have unselfishly devoted much time to the Transactions, and since the work is undoubtedly exceedingly important for our mathematical community. I did not escape permanently, however, for I was doing the equivalent of editorial criticism on the National Research Fellowship Board from 1924 to 1936, and have been an editor of the Carus Mathematical Monographs from 1924 on, and chairman of the editorial committee of the University of Chicago Science Series from 1929.

The Fellowship Board was a fine group of men, consisting, when I first became a member, of Simon Flexner, Chairman, K. T. Compton and Mendenhall and Millikan in physics, Johnston and Keyes and Kohler in chemistry, Birkhoff and Bliss and Veblen in mathematics. Johnston and Mendenhall later withdrew and were replaced by others. I often wondered how some of these men found the time for the conscientious work which they did on the Board. Our choices were criticized severely in particular cases, sometimes justly, sometimes not. At one time the three groups examined independently their list of past appointees, marking those who had distinguished themselves as Fellows A, those who would again be appointed B, and the failures C. When they compared notes the number of C's was in each field about ten percent of the total in that field. This seems to me an excellent record, and I wish that each of our critics might have an opportunity to try to equal or surpass it. But the effort to classify people according to their ability, to say that one is better than another, is distasteful to me, and in editorial work it seems to me a much greater pleasure to try to find the strong points of a paper than to criticize its deficiencies. In general I feel sure that our mathematical community will be more effective and much happier if we cultivate whenever possible an appreciative and cooperative point of view rather than a critically comparative one.

9. Philosophy of graduate work. When Mason was president of the University of Chicago our Department was under great pressure. We had too many students. He repeatedly urged us to restrict enrollment by setting higher standards for admission. I have always felt that this is a mistaken policy. Experience has shown that it is impossible in many cases to foretell an entering student's future. Some of the inexperienced ones develop real power, and some of the brilliant ones fade away. Many who are not qualified to take a higher degree get a real thrill and a new impetus for their teaching work out of the advanced courses. I therefore favor keeping graduate lecture courses open to all who have a reasonable justification for their desire to listen in. The time when distinctions should be made is when the student applies for candidacy for a higher degree, after he has given his instructors some indications of what he can be expected to

do. The candidates for higher degrees are the ones who take the time, not the listeners in lecture courses.

At the present time there are strong influences at work to cheapen higher degrees in mathematics, as also in other subjects. I hope that our standards can be preserved. In our Department we have been unwilling that a candidate for the master's degree should divide his requirement between mathematics and some other subject, because if he does so he can present for his degree only the most elementary graduate material in either field. We have insisted on a thesis containing if possible some new result of an elementary sort. The purpose of this is to give the student his first introduction to methods of research and to train him in mathematical writing. I advise students to take master's degrees on the way to their doctorates because of the great value to them of this preliminary training. We keep a record of the performance of a student in attaining his master's degree and use it as one of the aids to guide us in determining whether or not he should be recommended for candidacy for the doctor's degree. Thus we can avoid the necessity of preliminary examinations for the doctor's degree. I am much opposed to such preliminary examinations because they take much time and tend to place the emphasis on courses rather than research for both students and faculty. The sooner a candidate for the doctor's degree can start his research the better it will be for everybody.

It seems to me that there is widespread misunderstanding of the significance of doctor's degrees in mathematics. The comment is often made that the purpose of such a degree is to train students for research in mathematics, and that the success of the degree is doubtful because most of those who attain it do not afterward do mathematical research. My own feeling about our higher degrees is quite different. The real purpose of graduate work in mathematics, or any other subject, is to train the student to recognize what men call the truth, and to give him what is usually his first experience in searching out the truth in some special field and recording his impressions. Such a training is invaluable for teaching, or business, or whatever activity may claim the student's future interest.

Not long ago a committee of the Mathematical Association of America published a report in the AMERICAN MATHEMATICAL MONTHLY advocating the inauguration of a higher degree for teachers not emphasizing training in research. I should hesitate to believe that research experience is not as valuable for teaching as for other types of activity. Of the ten members of the committee nine have doctor's degrees in mathematics, and five are doctors of our own Department. Some have been active in mathematical research, and others have devoted themselves to other equally important duties in their respective environments. I would venture the opinion that the training for the doctor's degrees held by these committee members has in each case been an important factor in establishing them in the positions of usefulness and influence which they now hold. The suggestion of a higher degree for prospective college teachers, comparable with the doctor's degree but not requiring a research thesis, is a very old one,

as old as my experience in university work. I think that some one who has confidence in such degrees should try to establish them. But they should not be called doctor's degrees, since this title has long implied research.

In a recent report R. G. D. Richardson has tabulated the research output of doctors of philosophy in mathematics in the United States, and has given comparative statistics for various American universities. His figures show among other things that many of our doctors do no research after they receive the doctorate; that a considerable number continue to publish in limited amount; and that a relatively small number continue to be active in mathematical research. The report itself is purely statistical, but it has been the basis for criticisms of the training of doctors in mathematics in this country which seem to me unjustified. The fact that many doctors do no mathematical research after receiving their degrees is quite in accord with the purpose of such degrees as described in a preceding paragraph above, provided that the doctors apply in whatever may be their chosen activity the spirit of the investigator. It is not easy to measure success in this respect, but it is clear that our doctors as a group have been given relatively very great and very important educational responsibilities in the universities of the country. Those of us who have devoted our lives to the intimate study of mathematics are of course greatly interested in the relatively small number of doctors who turn out to be continuously interested and successful in research. The percentage of doctors in this group will be small in mathematics, as in any other subject. To insure adequate renewals in the ranks of such scholars it is necessary that our population of graduate students of mathematics shall always be as large as is reasonable in view of the demand for trained mathematicians in our educational and other institutions. Furthermore for the continuation of the activity and interest of this relatively small group of enthusiasts the presence of the much larger number of mathematicians who do some research themselves and who have high appreciation of it in others is essential. No great mathematical school of the past, with its heroes to whom we often ascribe too exclusively its achievements, ever flourished without its public of well-trained and appreciative listeners. Thus it seems to me that doctors of all the types described in Richardson's statistics are essential to the well being of our American mathematical school, and that criticisms of our doctorate based upon the figures given in his paper are not well founded.

I would make one final comment on Richardson's report relative to the doctors of our own Department. Their number is larger than that of any other mathematical Department, but the average of their annual publication output is lower there than in a number of other cases. The average of their total number of published pages per year is also the largest. If one is thinking only of mathematical research produced it is evident that magnitude of production can be attained either by a considerable number of workers each producing a limited amount, or by a smaller number of scholars each of whom is very active. At Chicago we have turned out men of both types and though our total productivity is large our average per doctor is relatively smaller. The discussion of

these questions impresses one, however, with the futility of attempts at measurement of the relative effectiveness of different departments. What is needed in every case is a group of men devoted to mathematics and ambitious to make the most effective use of their own personalities and interests in research and teaching. In every environment it is important to stimulate the strong graduate student by the challenge of his research thesis, but it is still more important to impress upon him the significance of a first independently conceived research paper following the doctorate. The preparation and publication of such a paper is the first conclusive evidence of true interest in scholarship.

It is clear from the preceding paragraphs that different individuals and different groups have different interests and different ideas about the training of students, and that different methods in research and teaching may achieve comparable successes. For those who have demonstrated their own effectiveness the most important things which a university can offer are freedom and encouraging recognition. There is nothing more discouraging in university work than to be continually told that methods and ideals of the past have for the most part been seriously wrong, even if one knows from his own experience that in his own department of learning the charge is not well justified. Unfortunately the psychology of our most enthusiastic students of the theory of education at the present time seems to be that of the pessimist and critic, as one may verify by an examination of the addresses on almost any recent program of speakers on educational theory.

UNIQUE ARRANGEMENTS OF POINTS ON A SPHERE

L. L. WHYTE, London

It is over seventy years since Sohncke established the theory of the 65 (infinite) *regular rectilinear point systems* in three-space, while the theory of the (finite) *unique spherical point systems*, apart from the special cases of the regular and semi-regular polyhedral arrangements, is still at an early stage.

The reasons for the relative neglect of spherical arrangements are clear. Until this century there was no adequate stimulus to explore this field beyond these special cases, and the other unique arrangements cannot be treated by established methods such as group theory.

But once an unexplored territory is known to exist, the challenge sooner or later becomes irresistible, and it seems that this is now happening. The new field of enquiry may be defined as *the theory of spherical point systems, i.e., of those arrangements of a finite number of points on a sphere in three-space which are distinguished by some extremal property*. Since 1904 more than a dozen workers

[1] have examined problems lying within this area, all but two during the last ten years.

The present paper reviews this work in order to call attention to a group of unsolved problems which may have applications to physics. Until recently the angular relations of sets of particles have been relatively neglected in theories of the structure of matter. But a systematic analysis of the angular relations of unique systems of N points (perhaps representing the motions and stable equilibrium patterns of nucleons) may guide the development of physical theory in the future, just as the study of the linear relations of pairs of points (representing the motions of atoms or massive bodies) did in the past. In particular, the analysis of the angular structure of unique spherical shells of point particles may provide a first step towards the theory of the atomic nucleus, just as the study of regular linear arrays of identical atomic groups led towards the complete theory of crystal structure.

The spherical arrangements of points determined by the vertices of polyhedra having all their vertices equivalent have been studied from about 500 B.C. onwards. But it seems that no consideration [2] was given to unique spherical arrangements of N points determined by extremal values of some function of the arrangement until 1904 A.D., when J. J. Thomson [3] sought to determine the stable equilibrium patterns of N classical electrons ($1<N<100$) constrained to move on the surface of a sphere while repelling each other by the inverse square law (here called the Thomson problem). Thomson found this problem intractable for spherical arrangements, and used approximate methods to minimize the total electrostatic energy for plane rotating rings of electrons. His papers are of historical interest, for he showed how the existence of unique stable rings and shells might follow from simple assumptions, but his results were restricted to plane rings. This is not surprising, for the solutions of the Thomson problem even for relatively low values (*e.g.* $N=9$, 11, *etc.*) are still unknown, though an experiment simulating repelling particles might be carried out to determine the answer.

In 1912 L. Föppl [4], at Hilbert's suggestion, made a rigorous examination (on slightly different assumptions) of the stability of Thomson's arrangements, and published the earliest known study of spherical point systems, based on the model of a uniform sphere of positive electricity with N electrons free to move through it. Föppl obtained stable solutions for all values of N up to 8, and for 10 12, and 14, in which the electrons all lie on one spherical surface, but as these solutions result from rather special assumptions we shall not list them. From inspection of eight solutions Föppl suggested the following tentative rules:— (i) the presence of electrons at poles and of planar rings perpendicular to the polar axis; (ii) the twisting of the rings, so that each electron is opposite the centre of a space in neighbouring rings (see below); and (iii) the discrimination of special values of N (in his problem: 4, 6, 8, 12, 20, 22, 46, 94, 118, with periods of 24 or 48) permitting uniquely stable arrangements, as in the periodic table

and the nuclear series. These three suggested rules may constitute prototypes of general principles valid for a wider class of spherical point systems than those derivable from Föppl's assumptions.

The Thomson problem and its variants have recently attracted little interest owing to the success of theories of atomic structure (Bohr, Schrödinger, Pauli Exclusion Principle) in which the stability of the atom is no longer determined by minimizing the electrostatic potential energy of mutually repelling point particles. Moreover the view that dynamical, higher-dimensional, and statistical representations are necessary for both atomic and nuclear structure has drawn attention away from the angular structure of shells of point particles in three-space. Thus Föppl's work appears to be unknown to the workers who have recently obtained similar, but not identical, arrangements as solutions of a distinct mathematical problem.

In 1942 Fejes Tóth [5] (then Fejes) began to investigate the arrangements of N points on a unit sphere which maximize the least distance, d, between any pair (here called the Fejes problem). Fejes later described this problem as the densest packing of equal spherical caps on a sphere, and formulated it thus: What is the maximum number of points one can place on a unit sphere so that the distance between any two shall be greater than a fixed amount ϵ? (ϵ plays a role similar to that of a universal minimum length separating all nucleons. Another formulation which avoids the redundant reference to two lengths, ϵ (or d) and the radius of the sphere, is: How large can the angle of N equal non-overlapping cones be, if their vertices coincide? Or: How many cones of given angle can be packed around a point?)

Fejes proved (1943) that there always exist two points whose distance is not greater than $\{4-\csc^2 \frac{1}{6}\pi N/(N-2)\}^{1/2}$ and that for $N=3$, 4, 6, and 12 the limit is exact, the last three being the cases in which the faces of the regular polyhedron are equilateral triangles. (The Fejes limit is also approached as $N\to\infty$.) This inequality, which expresses the fact that the area of the unit sphere is always at least that of $2(N-2)$ equilateral spherical triangles of side d, was obtained independently by H. Hadwiger [6] (1942/3) and by Habicht and van der Waerden (1951, see below) using different methods. Possible solutions for $N=8$ and 20 were described by H. Rutishauser [7] (1942/3, published 1944/5) using the Fejes inequality.

A point of interest is that Föppl, Fejes, and Rutishauser all independently proved that the five regular polyhedra fall into two groups: the triangular polyhedra with 4, 6, and 12 vertices, which provide solutions of their respective extremal problems; and the polyhedra with three edges at each of 8 and 20 vertices, which do not. For example, the Fejes problem for $N=8$ is solved by the vertices of a *twisted cube or square antiprism with 16 equal edges* (two squares twisted so that the transverse joins are equal to the sides of the squares). This is an application to Fejes' problem of Föppl's second rule, as shown in the table on page 610.

We must now consider a physical principle which led to interest in spherical point systems from a more general point of view. Thomson had been concerned with the stability of spherical arrangements in classical electron theory, but a similar problem arises in *any* physical theory of the stability of a finite number of identical entities of *any* kind grouped around a centre, provided that these entities can to a first approximation be represented by points on a sphere. In an extensive class of systems (*e.g.* bubbles, certain symmetrical molecules, crystalline structures, *etc.*) stability is associated with geometrical symmetry, *i.e.*, the state of minimum potential energy possesses an additional symmetry element, and the corresponding type of asymmetry decreases as the system moves towards its stable equilibrium state [8]. But the stable equilibrium form of a system composed of N identical entities arranged around a centre cannot be *uniquely* distinguished by the presence of symmetry elements for values of N allowing no spherical arrangement in which all the points are equivalent. The criterion of symmetry must therefore be replaced by some more comprehensive extremal property, if all values of N are to be treated.

If the potential function is unknown, or if the use of a potential function is inappropriate, one can nevertheless assume that the equilibrium states of spherical arrangements of identical point particles are determined by *some* extremal property. We are thus led to ask, avoiding any premature restrictive assumptions: *What spherical arrangements possess extremal properties, of any kind?* A search undertaken in 1949/50 [9] showed that no comprehensive study had been made of such spherical point systems and that even for the special Thomson and Fejes problems no general solution had been found. Solutions up to $N=7$ were available for both (for Fejes' problem in a private Swiss communication), but these had been obtained by laborious trial and error methods.

Meanwhile a more powerful attack on Fejes' problem had begun without knowledge of any of the work cited, but confirming and extending the results previously established. In May 1950 B. L. van der Waerden lectured in Göttingen on results obtained jointly by himself and W. Habicht [10] on distributions of points on a sphere, including Fejes' upper limit for d, and the fact (then known in Switzerland, but unpublished) that the maximum for d is the same for $N=5$ and 6. They asked: Of what size must a sphere be to provide room for N points of minimal distance unity? Use was made of the graphs which arise by connecting all pairs which have exactly the distance unity.

K. Schütte was led by this lecture to solve the case $N=7$, and he and B. L. van der Waerden [11] published a paper strengthening the Fejes inequality by the use of concave functions, and giving solutions up to $N=9$, suspected solutions for eight higher values, and a table and graph of limits for the radius up to $N=32$. A new concept was introduced: the density of packing of the best arrangement measured against the density given by the limit of the Fejes inequality, reached only for $N=3$, 4, 6, and 12, which may be regarded as the "most perfect" arrangements.

The known arrangements yielding solutions of Fejes' problem are:

N	*Föppl's notation**	*Description*
4	(1, 3) or (2, 2)	Regular tetrahedron.
5	(1, 3, 1) or (3, 2)	No unique arrangement. Two at poles, three on equator with distances at least $\pi/2$.
6	(1, 4, 1) or (3, 3)	Regular octahedron.
7	(1, 3, 3)	4 equilateral spherical triangles have angles of 80°.
8	(4, 4)	Square antiprism, with 16 equal edges.
9	(3, 3, 3)	8 equilateral spherical triangles have angles of $\cos^{-1}(\frac{1}{4})$.
12	(1, 5, 5, 1) or (3, 3, 3, 3)	Regular icosahedron.
∞	—	∞ equilateral plane triangles: close packing of equal circles in a plane.

These results show that the Thomson and Fejes problems are not in general solved by identical arrangements. For example, for $N=7$ Föppl obtains (1, 5, 1) for his version of the Thomson problem.

Some other work may be mentioned. The Fejes' problem for $N=5$ was recently published [12] as an unsolved problem, and the solution given by C. S. Ogilvy and by L. Moser. Spherical configurations of particles which are in equilibrium under any law of force have been examined by J. Leech (to be published). Considerable work has been done on similar problems in higher dimensional space and on the surface and volume extremal properties of polyhedra inscribed in or about a sphere, but without further results relevant here.

Unsolved Problems

The following unsolved problems arise from this survey:

A. *Thomson's Problem.* General solution, or further special solutions.

B. *Thomson's Problem.* For more general, or arbitrary potential function.

C. *Fejes' Problem.* General solution, further special solutions, or stronger inequalities. Can Föppl's rules be formulated so that they are valid for all solutions of Fejes' problem?

D. *Other Extremals.* Determine the arrangements yielding extremals of other functions, *e.g.* maximizing the product of the distances of all pairs.

E. *General Theory of Spherical Point Systems in Three-Space.*

(i) What spherical arrangements possess extremal properties of any kind? What extremal problems lead to identical solutions?

(ii) What general rules, such as Föppl's, are valid for the extremals of any non-trivial function of spherical arrangements?

(iii) Can more general extremal functions (*e.g.* of arrangements on concentric spheres or of appropriately weighted points) be found which discriminate the two sets of numbers determining nuclear and chemical stability respectively (magic numbers, and periods in periodic table), as yielding the *lowest*

* Showing number of points at poles and in twisted rings with points opposite centres of spaces in neighbouring rings.

extremal values? (These lowest extremal values may represent highest density of packing and most stable arrangements.)

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ALGEBRAIC PROPERTIES OF N-VALUED PROPOSITIONAL CALCULI

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1. Introduction. This paper undertakes to investigate n-valued propositional calculi or composition algebras of functions over a discrete range to the range from several algebraic standpoints. First, the general algebra is considered in terms of its basic structure. Secondly, the binary algebra is studied with particular reference to the rings it may form under special choice of a pair of functions defined as sum and product. Thirdly, a partial ordering of the range being introduced, the lattice properties of the algebra and of the ideals in the ring structure are briefly dealt with. The concluding section is concerned with independent generating functions, analogues of the Sheffer stroke in the two-valued case, and includes as its principal result, the listing of all such commutative analogues for the three-valued case.

In each of the first three topics, a portion of the material is expository in nature. An attempt is made to unify and to some extent generalize the viewpoints of various investigators in this field, particularly B. A. Bernstein, Webb, Rosenbloom, and E. T. Bell. The discussion is algebraic throughout, terms peculiar to mathematical logic are used, when possible, only as alternatives to their algebraic counterparts. The algebraic terminology employed is restricted to that which is common to standard works in abstract algebra such as [3] [4] and [10] and definitions of concepts employed may be found in these texts. The standard terminology of mathematical logic is employed where practicable [8]. Where extended or new definitions are required they are given in the context.

2. The discrete composition algebra. An m-ary composition algebra consists of a set of functions of m variables defined over a range to the range and closed under the basic operation of composition, *i.e.*, if $f, f_1, f_2, \cdots, f_m$ are members of the set, then there is a (unique) member: $f(f_1, f_2, \cdots, f_m)$.*

If the range is finite and discrete, the total number of functions from the range to the range will in turn be finite and this totality will clearly form a composition algebra. If the range consists of n marks, this latter algebra may be termed an n-valued m-ary propositional calculus. It is defined up to naming and ordering of the marks, and for fixed m and n any two such algebras are isomorphic.

In the n-valued case any function may be given by its list of values corresponding to all possible combinations of the variables. A convenient choice for the mark of the range is the set of integers, $0, 1, \cdots, n-1$. In this case the table may be formed by writing in a column the numbers from 0 to n^m-1 to the base n adjoining sufficient zeros on the left of each number to give a total

* It will be noted that $f(\cdots\cdots)$ here denotes the *function* rather than its value at some point of the range. This notation is adopted in the belief that it will prove less confusing than the standard semicolon usage in the case of multiple composition. When a value of a function is used in the sequel, the change in meaning will be apparent from the context.

of m digits; the result will be all possible combinations of the variables, the corresponding values of the function may then be adjoined. Each function will thus be represented by one of the numbers from 0 to $n^{n^m}-1$, expressed to the base n with adjunction of zeros where necessary to give n^m digits and set beside the table of the variables. We may also consider the functions as representing the possible m-ary multiplication tables for all closed operations on n elements. The first concept is the common logical one of "truth tables"; the second is of considerable use in abstract algebraical considerations.

Composition is an $m+1$-ary operation, ordering a single element to each ordered set of $m+1$ elements. The algebra can also be described as containing n^{n^m} m-ary operations, one corresponding to each element. Among these there will be some whose result is independent of the functions appearing in certain of the positions of the ordered m-uple. Thus a constant function will be independent of all its arguments; a singulary function will depend on just (a particular) one of the arguments, *etc.* It has been shown that any function may be represented as a composition of functions which are binary or simpler in character. Thus for the most part we may confine our attention to the case $m=2$. Here we tread on familiar ground for the typical algebraic properties have long been named and studied.

We may, for example, speak of commutative and associative functions: $f(f_1, f_2)=f(f_2, f_1)$; $f(f_1, f(f_2, f_3))=f(f(f_1, f_2), f_3)$; f_1, f_2, f_3 all arbitrary.

Finally, it will be desirable to specify the "range variables," x and y, which may be thought of as the individual columns of the range table or their associated singulary functions: $x(f_1, f_2)=f_1$; $y(f_1, f_2)=f_2$ for all f_i, or again the (necessarily unique) pair of functions such that $f(x, y)=f$ for all f. In the m-ary case there will be m such functions: $x_1, x_2, \cdots, x_m$.

3. Algebraic properties:

a. Isomorphism and automorphism; conjugate functions: It is obvious from the definition that any biunique mapping on the set of marks to another set will introduce an isomorphism or abstract identity between the resulting complete composition algebras. In particular, a permutation of the marks will produce an automorphism. The group of these automorphisms is the symmetric group on n marks. Let $f_1, f_2, \cdots, f_k$ be the set of functions on which the automorphisms map f; then k divides $n!$ and there is a subgroup H_f of order $n!/k$ which leaves each of the f_i invariant. We say that f belongs to H_f. It is easy to construct functions belonging to any designated subgroup of the symmetric group. In particular, the range variables belong to the full group; the constant functions to the symmetric subgroup of order $(n-1)!$, *etc.* We shall be particularly interested in those functions belonging to the identity, $k=n!$. In the two-valued case the functions associated by the automorphisms are called duals. In the m-valued case, they will be called conjugates. To sum up:

THEOREM 1. *Each function possesses k conjugates where k divides $n!$ and belongs to a subgroup of order $n!/k$. There are functions in the algebra belonging to*

each subgroup of the symmetric group including the identity and the full group.

b. The algebra as a direct sum. It has been remarked that each function could be considered as a vector with components against the basic array. This vector may, in turn, be considered as an element in a direct sum algebra where each of the summands is the set of the marks. Any f orders a particular mark to any ordered m-tuple of the marks. Thus $f(f_1, f_2, \cdots, f_m)$ may be described as the result of the operation f on the elements $f_1, f_2, \cdots, f_m$ of the direct sum. This concept is of particular value in finding functions, $f, g, \cdots$ which, considered as operations, exhibit the algebra as possessing a particular algebraic structure.

For example, we may ask, "Under what operations is the binary two-valued algebra a ring?" The answer is: Under those operations derived by the direct sum from those under which two marks form a ring. There are, up to isomorphism, two such rings, one isomorphic to the ring of integers modulo 2, and a nil-product ring. By direct sum, the former gives the ring of Stone and Bernstein and its dual.

It will be convenient to use as an alternate to the term "finite complete composition algebra," the words "propositional calculus," and to designate the m-ary, n-valued case by $PC(n, m)$.

THEOREM 2. *Every binary propositional calculus contains at least one pair of functions, f^+ and $f^\times$, with respect to which the calculus is a ring.*

THEOREM 3. *Under no set of operations is any propositional calculus a division ring or, a fortiori, a field.*

Proofs. Theorem 2: The integers $0, 1, \cdots, n-1$ form a ring modulo n.
Theorem 3: The direct sums are not simple.

The strongest structure which can be expected is a semi-simple ring; this exists for any value of n. It is merely necessary to pick basic operations under which the marks form a semi-simple ring. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, choose basic operations under which the marks themselves are the direct sums of the Galois Fields with $p_i^{a_i}$ elements. The calculus is thus the direct sum of direct sums of fields, hence semi-simple.

THEOREM 4. *For any n, at least two operations exist under which the calculus is a semi-simple ring.*

c. Polynomial representations. In certain cases the functions of the algebra can be represented as polynomial functions in the range variables. Bernstein [2] has done this when n is a prime; there is a simple generalization to the case, $n = p^a$.

THEOREM 5. *If $n = p^a$, the $PC(m, n)$ is isomorphic to the set of polynomial functions with coefficients from the Galois field of n elements, $GF(n)$.*

Proof: Select, by the direct sum process, two operations, f^+ and $f^\times$ under which the marks form the $GF(n)$; using these basic operations form the set of polynomial functions on the range variables with the constant functions as coefficients. This set contains n^{n^m} unequal elements since $x_i^n = x_i$, and the theorems concerning the number of zeros of a polynomial with coefficients in a field are valid here. The one-one mapping to the functions of the statement of the theorem is evident and the isomorphism follows from the fact that the process of composition is precisely that of substituting functions for the range variables in the polynomials.

The last theorem furnishes, for applicable cases, an extremely simple proof of a general principle previously mentioned:

COROLLARY. *If $n = p^a$, any m-ary function can be represented as a composition of binary functions.*

Proof: The binary functions are the f^+ and $f^\times$ of the preceding theorem.

To illustrate the application of the polynomial representation, let us determine the possible implications in the three-valued case which satisfy the following requirements:

1) $p \cdot p \supset q \cdot \supset q$ where conjunction is to have its commonly accepted meaning of "minimum truth-value."

2) The implication should reduce to the standard two-valued function if the "intermediate" mark is dropped.

Let the marks in ascending order of truth value be 0, 1, and 2, and designate as sum and product the corresponding operations of the $GF(3)$. Conjugation is then represented by $C(x, y) = 2xy(xy+x+y+2)$. We desire $I(x, y) = ax^2y^2 + bx^2y + cxy^2 + dx^2 + ey^2 + fxy + gx + hy + j$ such that $I(C(x, I(x, y)), y) = 2$. Also the values of $I(x, y)$ must be such that $I(0, a) = 2$, $a = 0, 2$; $I(2, 0) = 0$, $I(2, 2) = 2$. Substituting and comparing coefficients, 20 possible functions result. The values for the cases not already determined may be tabulated:

x	y	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
1	0	0	0	2	0	2	2	2	2	2	0	0	2	0	0	0	2	2	0	0	2
1	1	2	2	0	0	2	2	0	0	2	0	0	2	2	2	0	2	2	2	2	0
1	2	2	0	0	0	2	0	2	0	0	2	0	2	0	2	2	2	0	2	0	2
2	1	2	0	2	2	2	0	2	0	1	2	0	0	1	0	0	1	2	1	2	0

If, further:

3) The value of $I(x, y) \neq 2$ if the truth value of x exceeds that of y, there remain: 2, 11, 13, 14, 15, 18. Of these, 13 does not satisfy:

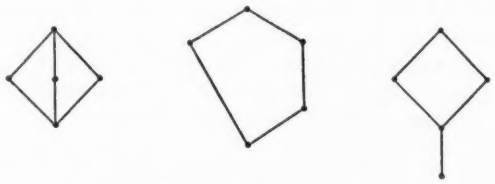
4) Implication is transitive. Finally, 14, 15, and 18 also satisfy:

5) The value of $I(x, y) = 2$ when the truth value of x is less than that of y. On intuitive grounds, a satisfactory case for the adoption of any of these three (or perhaps some of the others) might be made, depending on the individual interpretation of the mark 1. *Cf.* Webb [13].

d. Partial ordering on the marks: For any application of the propositional calculus to logic, there must exist a partial ordering relation on the marks, and, in general, a "true" or maximal mark and a "false" or minimal mark. This idea has already been used in the example. We now assume that the marks form a lattice. This is at once the simplest case with any actual generality and the most interesting one. In this case the propositional calculus also is a lattice, a direct sum of the lattices of the marks. We may define an implication relation, $\supset$, between elements of the calculus such that for two vectors, $V_1=(v_{11}, v_{12}, \cdots, v_{1n^m})$ and $V_2=(v_{21}, v_{22}, \cdots, v_{2n^m})$, $V_1 \supset V_2$ if and only if $v_{1j} \leqq v_{2j}$ for all j. Then in terms of the lattice representation, one element implies another if it is contained therein. The meet of two elements is the maximal element implying both and the join is the minimal element which both imply.

These lattices will be on n^m base lattices and will be n^m base lattices in length. Given any n, it is possible to exhibit the structure of all possible $PC(n, m)$ in a simple manner. There will always be the case where the marks are simply ordered and form a chain. The result is a direct sum of chains, a distributive lattice. This case has been studied in detail by P. Rosenbloom [9]. Their lattice properties are simple and readily noted. *E.g.*, they are complemented only for $n=2$ in which case they are Boolean Algebras.

For $n=2$ and 3, the chains are the only base lattices available. If $n=4$, the simple diamond is also possible; this form produces a Boolean algebra, a possibility when and only when n is a power of 2. For $n=5$, there are three possibilities besides the chain:



The first is modular but not distributive, complementation is not unique; the second is non-modular and non-uniquely complemented; the third is distributive but not complemented. The loss of the Boolean properties in the higher order calculi increases the difficulty of working with them and makes difficult or impossible the extension of theorems of the two-valued case. A strong case might be made for the diamond-based four valued logic as the "most reasonable" extension as in this case the well-known example is available as an endomorphic sub-lattice.

e. Ideals:

THEOREM 6. *The ideals of the $PC(n, m)$, written as a semi-simple ring, form a Boolean algebra on kn^m points, where k is the number of Galois Fields of which the ring of the marks is a direct sum. If $n=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$, $r \leqq k \leqq \sum a_i$.*

This is essentially a specialization of a well-known algebraic result, and the proof is immediate when the full ring is regarded as the direct sum of kn^m Galois Fields.

The lattice will have 2^{kn^m} elements compared to the n^{n^m} elements of the calculus. By definition, $2^k \leqq n$, the equality holding only when $n=2^s$ and $k=s$.

If $k=1$, $n=p^a$, a simple logical interpretation may be given to the ideals. Let 0 and α be the minimal and maximal marks with respect to truth value. All ideals in this ring are principal and generated by elements having the minimum number of zeros in their vector representation. One of these generators consists entirely of α's and zeros. (Let V be any generator, and consider αV^{n-1}). The ideal then consists of all elements implied by the generator.

4. Independent generating functions. In 1913, Sheffer [10] showed that there was a binary operation from which all other elements could be generated by composition from the basic elements. In 1925, Zylinski [15] showed that this operator and its dual were the only such binary operations for the two-valued case. In 1921, Post [7] gave an argument which produces such an operation for the n-valued case. Explicit derivations and definitions of other such elements were given by Webb [12], [14] and Martin [6].

Some general considerations concerning such elements follow:

a. If an independent generator is known, all its conjugates are independent generators.

b. Any independent generator has $n!$ conjugates. If a certain element is generated by a particular composition sequence, conjugation must yield the conjugate elements. Thus, if the generator did not possess its full complement of conjugates, it cannot generate those elements with $n!$ conjugates.

c. Any independent generator must generate all the singulary functions.

d. No independent generator can be commutative and associative. If $f(f(x, y), z)=f(x, f(y, z))$ and $f(x, y)=f(y, x)$, the sequence of composition could be represented as powers of x and y and any function would be equivalent to a form x^iy^k. Further $x^n=x$ since each row in the function table must have a period with respect to exponentiation which is a divisor of n. Thus only n^2 elements can be generated by such a function.

e. The value of a generator, f, a "double mark," say $x=y=b$, must not be b. If it were, the composition process would never produce a mark other than b in this place in the function table.

f. The function must be non-degenerately binary. (The last two remarks suffice to establish Zylinski's result that the Sheffer stroke and its dual are the only possible generators for the two-valued case. Of the four elements which pass requirement e), two are the negatives of the range variables.)

g. Further, no mark may be "trapped" within a set of marks. *E.g.*, if for $n>2$, the values of $f(x, y)$ corresponding to the pairs (a, a), (a, b), (b, a), (b, b) consist only of a's and b's, no other mark could be produced for the pairs (a, a) and (b, b).

With these remarks we will proceed to obtain all commutative single generators for the three-valued propositional calculus. There are, in all $3^6=729$ commutative functions. Applying requirement e), we have $2^3 3^3=216$ remaining, which reduce to 108 when g) is applied. Of these, six are eliminated by b) as they are conjugate two by two of three separate classes. It is interesting to note that one such class is the only one of the 20 which satisfies the requirements of solvability and is thus a commutative loop operation. *I.e.*, for these functions, f, and for all functions, a and b, there exists a unique function, c, such that $f(a, c)=b$.

Using the marks, 0, 1, 2, and abbreviating the basic table by reason of the commutativity, the last rejected elements are:

		α_1	α_2	β_1	β_2	γ_1	γ_2
0	0	1	2	1	2	1	2
1	1	2	0	2	0	2	0
2	2	0	1	0	1	0	1
0	1	0	1	1	0	2	2
0	2	2	0	0	2	1	1
1	2	1	2	2	1	0	0

Here α_l are the loop operations. The rejection of α, β, γ provides a counter example to a plausible conjecture that all elements whose values on the "double lines," (a, a), represented a permutation of the marks such that no element was unaffected, are generators. All of these elements are of this type. All previously known generators were of this type.

The permutations performed on base elements and vectors to obtain the conjugates will be designated: 1. Identity; 2. Interchange 1 and 2; 3. Add 1 modulo 3; 4. Interchange 0 and 1; 5. Add 2 modulo 3; 6. Interchange 0 and 2. The 102 remaining elements constitute 17 classes. For brevity, we list one example of each class. In the sequel the elements will be distinguished by the number in the following list and a subscript indicating which permutation produces the function in question.

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	0	0	0	0	0	0	0	0	2	2	2	2	2	2	2	2
2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	2	2	2	2	2	2	2	2	0	0	0	0	0	0	1	1
0	2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	0	1
1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	0

For example, 14_2 is:

0	2	1	0	0	2,	or:	0	1	2	0	0	1.
0	2	1	2	1	1		0	1	2	1	2	2
2	1	0	0	2	2		2	0	1	2	0	2

Further notation: S will denote an arbitrary element chosen from 10_1 to 17_1, inclusive; P, one of 1_1 to 9_1. The letters x and y will denote the first and second basic elements, $\bar{x}$ and $\bar{\bar{x}}$ denote the first and second "negations" of x, the elements obtained from x by adding 1 and 2 respectively, modulo 3 to the values of x. In general, $S(x, x) = \bar{x}$; $S(\bar{x}, \bar{x}) = \bar{\bar{x}}$. Thus, all the S operations generate both negations. Let $S(\bar{x}, S) = S_x$, $S(\bar{\bar{x}}, S_x) = S_{x^2}$, $S(x, S_{x^2}) = S_{x^3}$.

We shall establish the following:

THEOREM 7. *Classes* 1, 2, 3, 5, 6, *and* 8–17 *inclusive are generators and these* 90 *operations are the sole commutative generators for the three-valued case.*

In the proof, Webb's generators will furnish the initial information, and other functions will successively be established as generating functions which have previously been proved to be generators. It will then be shown that 4 and 7 do not meet requirement c).

In following any of the proofs which do not depend simply on negation, it will be necessary to rewrite the functions in terms of the full truth table rather than employing the abbreviated table used in the definition. This is necessary because while S is commutative, S_x is in general not commutative.

1) 10_1 and 17_1 are Webb's generators.

2) $\overline{16}_1 = 17_6$; thus the 16 class consists of generators.

3) $\overline{15_1(15_{1^x}, 15_{1^y})} = 11_3$.
$15_1(15_{1^x}, 15_{1^{y^2}}) = 12_4$, $\overline{12}_4 = 13_1$.
$11_3(11_3, 13_1) = 10_4$; and 15 is a generator.

4) $\overline{13_1(13_{1^x}, 13_{1^y})} = 15_5$; therefore 13 is a generator.

5) $\overline{12}_1 = 13_4$; and 12 is a generator.

6) $\overline{11_1(11_{1^x}, 11_{1^y})} = 13_3$; thus 11 is a generator.

7) $\overline{14}_1 = 11_6$.

8) Let $\alpha(x) = \overline{P(x, x)}$; $x_p^n = P(x, x_p^{n-1})$, $x_p^0 = x$. The subscript will be omitted when no ambiguity will result. Define $(x_p^n)^m$ as $P(x_p^n, (x_p^n)^{m-1})$, and $(x_p^n)^0 = x^n$. This notation is essentially that of Webb. Let $P = 1_1$, $x^4 = \bar{x}$ and $\alpha(x)$ is thus generated by P. Then: $1_1(1_1, \alpha(1_1(\bar{\bar{x}}, \bar{\bar{y}}))) = 17_4$. Therefore 1 is a generator.

9) Let P be 2_1, denote $(x^1)^2$ by $\beta(x)$ and $2_1(\beta, \beta)$ by $\gamma(x)$. Then $\bar{x} = 2_1(x, \gamma)$ so 2_1 generates $\alpha(x)$. Now $2_1(2_1, \alpha(2_1(\bar{\bar{x}}, \bar{\bar{y}}))) = 12_2$ and 2 is a generator.

10) Let $P = 3_1$; $x^4 = \bar{x}$; $3(3_1, \alpha(3_1(\bar{\bar{x}}, \bar{\bar{y}}))) = 10_2$ and 3 is a generator.

11) Let $P = 5_1$; $x^4 = \bar{\bar{x}}$; $5_1(\overline{5}_1, 5_1(\bar{x}, \bar{y})) = 15_3$ and 5 is a generator.

12) Let $P = 6_1$; $x^7 = \bar{x}$; $\overline{6}_1(6_1, x^1(6(\bar{\bar{x}}, \bar{\bar{y}}))) = 16_3$ and 6 is a generator.

13) Let $P = 8_1$; $8_1((x^1)^1, x^6) = \bar{\bar{x}}$; $\overline{8}_1(8_1, x^1(8_1(\bar{\bar{x}}, \bar{\bar{y}}))) = 10_1$ therefore 8 is a generator.

14) Let $P = 9_1$; $9_1(x, 9_1(x^2, x^3)) = \bar{x}$; $\overline{9}_1(9_1, x^1(9_1(\bar{\bar{x}}, \bar{\bar{y}}))) = 10_1$ and 9 is a generator.

15) To prove that 4 is not a generator, we shall show that no possible form beginning with the basic form (012) will result in (001).

a. To generate (001) by 4_1 requires at least one of: (110), (120), (210), or (220).

b. To generate one of the forms in a) requires either forms already in a) or one of the following: (002), (201), (202), (101), (102).

c. To generate any of the preceding, requires either one of those already listed or one of: (010), (020).

d. To generate (010) or (020) requires one of the forms already listed; the chain is then closed and (001) cannot be produced.

16) Similarly 7 is not a generator. We may vary the argument and demonstrate that from the basic form (012), 7_1 will generate only (000), (011), (012), (022), (100), (111), (122), (200), (211), and (222); and the chain is closed. Thus (001) will not be generated.

Some general conclusions may be mentioned in addition to the hypothesis discussed following the rejection of α, β and γ. First, there are in general, independent generators whose values on the double lines are not simple permutations of the marks; 42 such have just been produced. Second, there is no simple rule by which independent generators may be distinguished by their table of values; a comparison of 4 and 7 with some of the others will make this apparent. Third, the number of generators is not, at least absolutely speaking, small. From the two cases available no intelligent conjecture as to the number of such generators for the n-valued case is possible but it is interesting to note that one-fourth of the commutative functions for the two-valued case satisfy the requirements while roughly one-eighth of those in the three-valued case are generators. Recently N. M. Martin has undertaken the task of continuing the results given in this section by determining the non-commutative Sheffer strokes also for the three-valued case.

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MATHEMATICAL NOTES

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A NEW PROOF OF THE COMPLETENESS OF THE HERMITE FUNCTIONS

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The purpose of this note is to give a short proof of the completeness in L_2 of the Hermite functions.

We define the nth Hermite function to be

$$\psi_n(x) = (-)^n e^{\frac{1}{2}x^2}(2^n n!\sqrt{\pi})^{-\frac{1}{2}} \times \frac{d^n}{dx^n} e^{-x^2}, \qquad n = 0, 1, 2, \cdots .$$

As is well known, these functions form an orthonormal set over the infinite line, and

$$(1) \qquad e^{\frac{1}{2}x^2}e^{-(x-\lambda)^2} = \sum_{n=0}^{\infty} \lambda^n \left(\frac{2^n\sqrt{\pi}}{n!}\right)^{\frac{1}{2}} \psi_n(x), \qquad (-\infty < \lambda < \infty, -\infty < x < \infty).$$

With the aid of (1) and the Riesz-Fischer theorem, it is easy to show that

$$(2) \qquad e^{\frac{1}{2}x^2}e^{-(x-\lambda)^2} \sim \sum_{n=0}^{\infty} \lambda^n \left(\frac{2^n\sqrt{\pi}}{n!}\right)^{\frac{1}{2}} \psi_n(x), \qquad (-\infty < \lambda < \infty, -\infty < x < \infty).$$

We now set $2\lambda = \mu$ and $\beta_n = e^{2\lambda^2}\lambda^n(2^n\sqrt{\pi}/n!)^{\frac{1}{2}}$ and rewrite (2) in the following form:

$$e^{-\frac{1}{2}(x-\mu)^2} \sim \sum_{n=0}^{\infty} \beta_n \psi_n(x), \qquad (-\infty < \mu < \infty, -\infty < x < +\infty).$$

Thus if $f(x)$ is any member of L_2 which can be represented in the form

$$f(x) \sim \sum_{0}^{m} A_n e^{-\frac{1}{2}(x-\mu_n)^2}, \qquad m < +\infty,$$

then we can also write

$$f(x) \sim \sum_0^\infty B_n \psi_n(x).$$

To prove the completeness of the Hermite functions it is then sufficient to prove that the set of all translations of $e^{-\frac{1}{2}x^2}$ is complete in L_2, and this is easily done by means of the following theorem, whose proof does not depend on the completeness of the Hermite functions.

THEOREM (Wiener*) *Let $f(x)$ be a member of L_2. The set of all its translations will be complete in L_2 if and only if the real zeros of*

$$G(u) = \text{l.i.m.}_{\beta\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\beta}^{\beta} e^{iux} f(x)\,dx$$

form a set of measure zero.

For if we take $f = e^{-\frac{1}{2}x^2}$ then $G = e^{-\frac{1}{2}u^2}$, Wiener's theorem holds, and the proof is complete.

THE APPROXIMATION OF RADICALS BY RATIONAL MEANS

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1. **Introduction.** In this paper we explain a method of approximating radicals of any order. Since a general statement is unavoidably complicated, the particular case of fourth roots, written out in full, serves to illustrate the general argument.

If a_0, b_0, c_0, d_0 are positive, and

$$a_{n+1} = \frac{a_n + b_n + c_n + d_n}{4},$$

$$b_{n+1} = \frac{\dfrac{a_nb_n + a_nc_n + a_nd_n + b_nc_n + b_nd_n + c_nd_n}{6}}{\dfrac{a_n + b_n + c_n + d_n}{4}},$$

$$c_{n+1} = \frac{\dfrac{b_nc_nd_n + a_nc_nd_n + a_nb_nd_n + a_nb_nc_n}{4}}{\dfrac{a_nb_n + a_nc_n + a_nd_n + b_nc_n + b_nd_n + c_nd_n}{6}},$$

$$d_{n+1} = \frac{a_nb_nc_nd_n}{\dfrac{b_nc_nd_n + a_nc_nd_n + a_nb_nd_n + a_nb_nc_n}{4}}, \qquad n = 0, 1, 2, \cdots,$$

* Wiener, N., Tauberian theorems, Annals of Math., 1932.

we prove that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = (a_0b_0c_0d_0)^{1/4}.$$

2. **Proof.** The idea of the proof is this: Obviously

$$a_{n+1}b_{n+1}c_{n+1}d_{n+1} = a_nb_nc_nd_n,$$

whence

$$a_nb_nc_nd_n = a_0b_0c_0d_0, \qquad n = 1, 2, \cdots.$$

It is now necessary to prove that a_n, b_n, c_n, d_n tend to the same limit. We do this, first by proving [1, 2],

$$a_{n+1} \geqq b_{n+1} \geqq c_{n+1} \geqq d_{n+1}, \qquad n = 0, 1, 2, \cdots;$$

whence it is easily deduced that

$$a_{n+1} \leqq a_n, \qquad d_{n+1} \geqq d_n, \qquad n = 1, 2, \cdots.$$

Then we prove that $\lim_{n\to\infty}(a_n - d_n) = 0$. The result then follows.

First since

$$a_{n+1} - b_{n+1} = \frac{\sum (b_n + c_n + d_n - a_n - c_n - d_n)^2}{12\sum a_n},$$

$$b_{n+1} - c_{n+1} = \frac{\sum (b_nc_n + b_nd_n + c_nd_n - a_nc_n - a_nd_n - c_nd_n)^2}{6(\sum a_n)(\sum a_nb_n)},$$

$$c_{n+1} - d_{n+1} = \frac{\sum (b_nc_nd_n - a_nc_nd_n)^2}{2(\sum a_nb_n)(\sum a_nb_nc_n)},$$

therefore $a_{n+1} \geqq b_{n+1}$, $b_{n+1} \geqq c_{n+1}$, $c_{n+1} \geqq d_{n+1}$; that is

$$a_n \geqq b_n \geqq c_n \geqq d_n, \qquad n = 1, 2, \cdots.$$

Hence

$$a_{n+1} = \frac{a_n + b_n + c_n + d_n}{4} \leqq a_n,$$

$$d_{n+1} = \frac{4}{\dfrac{1}{a_n} + \dfrac{1}{b_n} + \dfrac{1}{c_n} + \dfrac{1}{d_n}} \geqq d_n, \qquad n = 1, 2, \cdots.$$

Again, since

$$a_{n+1} - d_{n+1} = \frac{\sum a_nb_n(c_n - d_n)^2}{4\sum a_nb_nc_n},$$

it follows that

$$a_{n+1} - d_{n+1} \leqq \frac{\sum a_n b_n \mid c_n - d_n \mid}{4\sum a_n b_n c_n} \cdot (a_n - d_n)$$

$$= \frac{-3b_n c_n d_n - a_n c_n d_n + a_n b_n d_n + 3a_n b_n c_n}{4(b_n c_n d_n + a_n c_n d_n + a_n b_n d_n + a_n b_n c_n)} \cdot (a_n - d_n). \tag{1}$$

For fifth roots, these statements would be

$$a_{n+1} - e_{n+1} = \frac{\sum a_n b_n c_n (d_n - e_n)^2}{5\sum a_n b_n c_n d_n},$$

$$a_{n+1} - e_{n+1} \leqq \frac{-4b_n c_n d_n e_n - 2a_n c_n d_n e_n - 0a_n b_n d_n e_n + 2a_n b_n c_n e_n + 4a_n b_n c_n d_n}{5(b_n c_n d_n e_n + a_n c_n d_n e_n + a_n b_n d_n e_n + a_n b_n c_n e_n + a_n b_n c_n d_n} \cdot (a_n - e_n).$$

From (1), the crude inequality

$$a_{n+1} - d_{n+1} \leqq \tfrac{3}{4}(a_n - d_n)$$

is sufficient for our purpose.

References

1. These inequalities were stated by Newton.

2. For other proofs see G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Cambridge, 1934, pp. 52, 104.

CUBIC EQUATIONS IN GROUPS

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In groups, an analogue of the nth degree equation is the equation, $UA_{n-1}UA_{n-2}\cdots A_2UA_1U=I$. This paper concerns the general solution to the above equation when $n=3$. By a general solution, I mean: an expression containing only the coefficients A_1 to A_{n-1} and the operations of multiplication, division, and root-extraction (assuming that the latter has meaning), which, when substituted for U in the equation, reduces the equation to an identity. The quadratic equation $UA_1U=I$ has the general solution $U=\sqrt{A_1^{-1}}$.

I shall now prove that the cubic equation $UA_2UA_1U=I$ has no general solution in the sense described above. Suppose that there were an expression that would reduce the cubic equation to an identity when substituted for U. Since, in this expression, there must be only a finite number of root extractions, there must be, at one point, a highest root, say the Nth, extracted. It follows that in any group, in which all root extractions of degree equal to or less than the Nth have meaning, all cubic equations in the group have solutions. I will now show that for any N there is a group in which all root extractions of degree equal to or less than N have meaning in the group, but such that there is a cubic equation in it with no solution and hence the absurdity of a general solution.

If P is the smallest divisor ($\neq 1$) of the order h of the group H, then the equation $U^k = q$ has a solution for all q in H if k is less than P. In fact U is in the cyclic subgroup generated by q.

Now I define a group* by the following relations between its generators α and β, with $t^r \equiv 1 \pmod{s}$:

$$(1)\ \alpha^r = I \qquad (2)\ \beta^s = I \qquad (3)\ \beta\alpha = \alpha\beta^t.$$

From relation (3) we obtain the relation $b^m\alpha^n = \alpha^n\beta^{mt^n}$. The reader may easily verify that any product of the α's and β's can be represented uniquely by the form $\alpha^m\beta^n$, and that these forms constitute a group of order rs.

In the above defined group, the cubic equation becomes

$$U\alpha^c\beta^d U\alpha^e\beta^f U = I,$$

where $U = \alpha^z\beta^y$ is the unknown. By multiplying out, we obtain

$$\alpha^{3x+c+e}\beta^{y(t^{2x+c+e}+t^{x+e}+1)+dt^{2x+e}+ft^x+y} = I.$$

This is equivalent to the set of congruences

(i) $$3x + c + e \equiv 0 \pmod{r}.$$

(ii) $$y(t^{2x+e+c} + t^{x+e} + 1) + dt^{2x+e} + ft^x \equiv 0 \pmod{s}.$$

Now I shall prove the theorem by showing that, for any N, I can find numbers r, s, t, c, d, e, and f, defining the aforementioned group, such that the smallest divisor, greater than one, of rs is also greater than N and further that the congruences (i) and (ii) will have no simultaneous solution.

Let s be a prime of the form $12n-1$ and let h be a primitive root of s. Also let $t \equiv h^2 \pmod{s}$; $r = (s-1)/2 = 6n-1$, and let $d = 0$ and $f = 1$.

If the congruence $t^A + t^B + 1 \equiv 0 \pmod{s}$ holds for some integers A and B, then the congruences (i) and (ii) have no simultaneous solution for some integer values of c and e.

Proof: Let $A + B \equiv e \pmod{r}$ and $2A - B \equiv c \pmod{r}$, so $3A \equiv c + e \pmod{r}$. And if congruence (i) is to hold $-3x \equiv 3A \pmod{r}$, or $-x \equiv A \pmod{r}$; but $2x + c + e \equiv -x \equiv A \pmod{r}$ by congruence (i), and $x + e \equiv A + B - A \equiv B \bmod r$, so $t^A + t^B + 1 \equiv t^{2x+e+c} + t^{x+e} + 1 \equiv 0 \pmod{s}$, and so the congruence (ii) has no solution. The congruence $t^A + t^B + 1 \equiv 0 \pmod{s}$ has a solution if the congruence $h^Q + h^W + 1 \equiv 0 \pmod{s}$ has a solution where both Q and W are even.

In the solutions of the congruence $h^Q + h^W + 1 \equiv 0 \pmod{s}$ such that $Q \geqq W$ (where Q and W need not be even) all the reduced residues modulo s occur exactly once as either h^Q and/or h^W except the residue -1 since $h^W \equiv -1 \pmod{s}$ would imply that $h^Q \equiv 0 \pmod{s}$ which is absurd. Since h is a primitive root of s all of the reduced residues modulo s would appear once as either Q and/or W except $(s-1)/2$ since $h^{(s-1)/2} \equiv -1 \pmod{s}$.

* An interesting generalization of this group can be found in Zassenhaus "The Theory of Groups"; Chelsea Pub. Co., 1949, p. 98.

Now consider the set E of numbers $1, 2, 3, 4 \cdots s-1$ with the number $(s-1)/2$ removed. Since $(s-1)/2$ is odd, E has more even numbers in it than odd numbers. Also each solution of $h^Q+h^W+1\equiv 0 \pmod{s}$ is a pair of numbers (Q, W) taken from the set E, and no two pairs have the same number in common. Since E contains more even numbers than odd numbers, it follows that some solution pair (Q, W) is a pair of even numbers, and so the congruence $t^A+r^B+1\equiv 0 \pmod{s}$ has a solution.

It remains to be proved that the smallest divisor $(\neq 1)$ of the order of the group I have defined can be made greater than any preassigned number.

By a well known theorem the number of primes in any reduced residue class is infinite. And so, if we make $s\equiv -1 \pmod{3\cdot 4\cdot 5\cdot 7\cdot 11\cdot 13 \cdots K}$ where K is an arbitrarily large prime, then the smallest divisor of the order $s(s-1)/2$ of the group is greater than K, and the theorem is proved.

It is interesting to notice that in the group defined by the relations,

$$(1)\quad \alpha^7 = I, \qquad (2)\quad \beta^{43} = I, \qquad (3)\quad \beta\alpha = \alpha\beta^4,$$

the cubic equation $U\alpha U\alpha^4\beta U = I$ has no solution, but it can be verified by trial and error that all *fourth* degree equations in this group have solutions.

CLASSROOM NOTES

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ON THE ENVELOPE OF CURVES GIVEN IN PARAMETRIC FORM

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In a recent calculus class, the writer asked for the envelope of the family of ellipses in standard position, the sum of whose major and minor axes is 2. One student argued as follows. The parametric equations of the ellipse are $x=a\cos\theta$, $y=b\sin\theta$ or equivalently, $a=x/\cos\theta$, $b=y/\sin\theta$. It is required that $a+b=1$; hence $x/\cos\theta+y/\sin\theta=1$. If this last equation is differentiated with respect to θ and θ eliminated, there results $x^{2/3}+y^{2/3}=1$, which is correct.

At first it appeared that the student had had a striking piece of good fortune, since what he had actually done was to find the envelope of the family of lines on which the axes cut off a segment of unit length. The two families have the same envelope. An explanation, however, of his good luck is to be found in a fact well known to many persons, especially to those mathematicians who work with the calculus of variations, where the subject comes up naturally, but apparently not widely familiar to mathematicians and teachers in general. The

writer has not seen more than a bare mention of it in any of the textbooks currently used for instruction in the calculus, and it is his opinion that such an excellent and instructive application of a number of the notions and processes of calculus is worth knowing.

What the student did is as follows. He had, replacing b by $(1-a)$, the parametric equations

$$x = a\cos\theta, \qquad y = (1-a)\sin\theta, \tag{1}$$

of a family of curves, where a is the parameter which distinguishes one curve of the family from the other, and θ is the "running parameter" of the individual curves. He interchanged the roles of these two parameters and found the envelope of a different family of curves without affecting the result.

The key to the situation lies in the condition for the envelope when the curves are given in parametric form:

$$x = f(\theta, a), \qquad y = g(\theta, a). \tag{1'}$$

Suppose that we eliminate the running parameter θ from equations (1') by solving for it from the first and substituting the result in the second. We have then for the equation of our family of curves in nonparametric form,

$$y = g(\theta(x, a), a), \tag{2}$$

where

$$x \equiv f(\theta(x, a), a). \tag{3}$$

We now obtain the condition for the envelope in the usual way by differentiating (2) with respect to the parameter a:

$$0 = g_1(\theta(x, a), a)\frac{\partial\theta}{\partial a} + g_2(\theta(x, a), a).$$

To obtain $\partial\theta/\partial a$ we differentiate (3) implicitly:

$$0 \equiv f_1(\theta(x, a), a)\frac{\partial\theta}{\partial a} + f_2(\theta(x, a), a).$$

Eliminating $\partial\theta/\partial a$ between the last two equations, we have the condition

$$\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix} = \frac{\partial(f, g)}{\partial(\theta, a)} = 0 \tag{4}$$

for the envelope, together with the equations (1'). In the case of the equations (1), $\partial(f, g)/\partial(\theta, a) = a - \cos^2\theta$, and substituting $a = \cos^2\theta$ in (1) gives $x = \cos^3\theta$, $y = \sin^3\theta$, the parametric equations of the four cusped hypocycloid.

The condition (4), being merely the vanishing of the Jacobian, remains unchanged if we reverse the roles of the two parameters and so the envelope of the families (1) is the same, whichever parameter plays which role. The point of view

taken here is, of course, the formal one. Whether either of the families has an envelope is in the general case quite another matter. As a matter of fact, in most of the more obvious examples, one of the families will have an envelope and the other not, as for instance the families $x=a \cos \theta$, $y=(\sin \theta)/a$ of ellipses or hyperbolas.

The derivation of the condition (4) for the envelope is a very good illustration of the use of the implicit methods and composite function methods of partial differentiation, and if rigor is sought, it is an excellent exercise in the use of the implicit function theorem. Equation (4) appears in the literature in numerous places, but we have not seen the dual role of the parameters mentioned. In Courant's *Differential and Integral Calculus*, (4) is stated without proof on page 174 of volume II. In fact, both of the examples contained in (1) are given as illustrations of envelopes, but the connection between them is not pointed out. Equation (4) is derived in de la Vallée Poussin's *Cours d'Analyse* on page 398 of volume II, and the equivalent condition for curves in three space in Picard's *Traité d'Analyse* on page 444 of volume I (1942 edition). Also on page 430 of volume I of Goursat-Hedrick's *Course in Mathematical Analysis* there is a discussion leading to the Jacobian from a similar starting point.

Actually we might well have anticipated that (4) would be the criterion for the envelope. For a typical family of curves, for instance the ellipses considered above, the envelope is the boundary of the region in the x, y plane into which the transformation (1) carries the a, θ plane. That is, the envelope is a locus of points x, y where equations (1) do not define implicitly a, θ as single valued functions. Thus it is not surprising that the Jacobian vanishes on the envelope.

INTRODUCTION OF COMPLEX NUMBERS AS VECTORS OF THE PLANE

L. Fuchs, Budapest, and T. Szele, Debrecen, Hungary

It is a well-known elementary fact that the complex number field is the only possible field extension of degree 2 of the field of the real numbers. This statement may be sharpened to the assertion that *if a (not necessarily commutative) associative algebra of order 2 over the real number field has no divisors of zero, then it is isomorphic to the field of complex numbers*. A vector space of dimension 2 over the real numbers has a simple geometric interpretation as the set of the vectors of the Euclidean plane, this fact being used several times for introducing the complex numbers. Hence it seems to be natural to raise the problem of proving the cited theorem in a purely elementary geometric way. Such a proof would help beginners to get a deeper insight into the kernel of this fact. Our present purpose is to give such a proof which seems to have some interest in itself and to be new.

In what follows let the vectors of the plane be denoted by bold-face and real numbers by Greek letters. We shall prove the following

THEOREM. *Suppose we have defined addition and multiplication of any two vectors of the plane as well as multiplication of a vector by a scalar (i.e., a real number) with the following properties*:

(1) *addition is vector-addition*; it is therefore commutative, associative, and unique subtraction is possible;

(2) *multiplication by scalars is performed as usual*; therefore we have $\alpha(\mathbf{a}+\mathbf{b})=\alpha\mathbf{a}+\alpha\mathbf{b}$, $(\alpha+\beta)\mathbf{a}=\alpha\mathbf{a}+\beta\mathbf{a}$, $(\alpha\beta)\mathbf{a}=\alpha(\beta\mathbf{a})$ for arbitrary real numbers α, β and vectors $\mathbf{a}$, $\mathbf{b}$;

(3) *multiplication of vectors is associative*: $\mathbf{a}(\mathbf{bc})=(\mathbf{ab})\mathbf{c}$;

(4) $(\alpha\mathbf{a})\mathbf{b}=\alpha(\mathbf{ab})=\mathbf{a}(\alpha\mathbf{b})$;

(5) *divisors of zero do not exist*: $\mathbf{ab}=0$ implies either $\mathbf{a}=0$ or $\mathbf{b}=0$;

(6) *distributivity holds*: $\mathbf{a}(\mathbf{b}+\mathbf{c})=\mathbf{ab}+\mathbf{ac}$ and $(\mathbf{b}+\mathbf{c})\mathbf{a}=\mathbf{ba}+\mathbf{ca}$.

Then, apart from an affine transformation, our vector system is just the set of all vectors of the complex plane with the well-known operations. In other words, if (1)–(6) *hold, then the vectors of the plane form a field isomorphic to the complex number field.*

Proof. Clearly, only the statement concerning multiplication needs a verification.

First we show that multiplication of vectors is commutative. Let us consider a vector $\mathbf{x}$ not parallel to its own square $\mathbf{x}^2$. Such an $\mathbf{x}$ certainly exists, for if $\mathbf{y}\neq 0$ is any vector parallel to $\mathbf{y}^2$, *i.e.* $\mathbf{y}^2=\lambda\mathbf{y}$ (plainly $\lambda\neq 0$), then each vector $\mathbf{x}\neq 0$ not parallel to $\mathbf{y}$ has the required property. In fact, if we had $\mathbf{x}^2=\mu\mathbf{x}$, then from the equations

$$(\mu\mathbf{y}-\mathbf{yx})\mathbf{x}=0 \quad \text{and} \quad \mathbf{y}(\lambda\mathbf{x}-\mathbf{yx})=0$$

we would get

$$\mu\mathbf{y}=\mathbf{yx}=\lambda\mathbf{x}$$

on account of (5). But this would mean that $\mathbf{x}$ and $\mathbf{y}$ are parallel, contrary to our hypothesis. Choosing a vector $\mathbf{x}$ with the prescribed property, it is evident that the vectors $\mathbf{x}$ and $\mathbf{x}^2$ span the plane. Hence each vector $\mathbf{a}$ may be represented (uniquely) in the form

$$\mathbf{a}=\alpha\mathbf{x}+\beta\mathbf{x}^2$$

with some real numbers α, β. For vectors of this type it is obvious, in view of the above axioms, that they commute. Therefore we obtain

$$\mathbf{ab}=\mathbf{ba}$$

for all vectors $\mathbf{a}$, $\mathbf{b}$.

For the square of the vector $\mathbf{a}=\alpha\mathbf{x}+\beta\mathbf{x}^2$ we have;

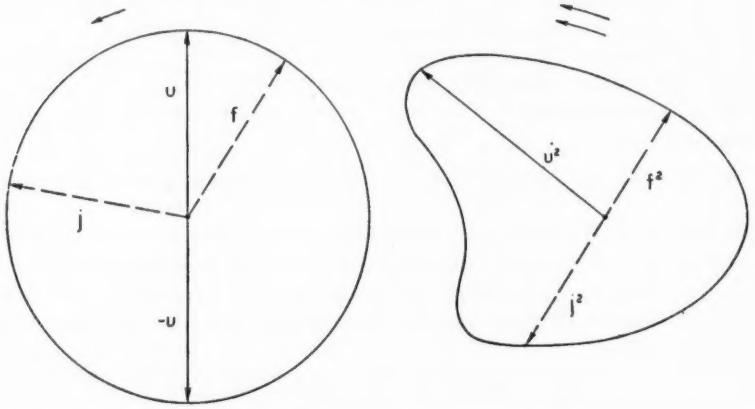
$$\mathbf{a}^2=\alpha^2\mathbf{x}^2+2\alpha\beta\mathbf{x}^3+\beta^2\mathbf{x}^4.$$

Hence it is seen that if we let $\mathbf{a}$ (*i.e.* α and β) vary a little, then $\mathbf{a}^2$ can vary but a little. Thus we get that if $\mathbf{a}$ varies continuously, then so does $\mathbf{a}^2$.

Now let $\mathbf{u}$ be an arbitrary unit vector and turn $\mathbf{u}$ through the angle π until it arrives at the position $-\mathbf{u}$. Meanwhile, according to what has been said above, the square vector has to rotate in one and the same direction, for, on the contrary, there would exist two non-parallel unit vectors $\mathbf{k}$ and $\mathbf{l}$ such that $\mathbf{k}^2=\lambda\mathbf{l}^2$ with $\lambda>0$. Hence, by commutativity, we have $(\mathbf{k}+\sqrt{\lambda}\mathbf{l})(\mathbf{k}-\sqrt{\lambda}\mathbf{l})=0$ and thus from (5) we are led to either of the equations

$$\mathbf{k}+\sqrt{\lambda}\,\mathbf{l}=0 \quad \text{and} \quad \mathbf{k}-\sqrt{\lambda}\,\mathbf{l}=0$$

contradicting the assumption on $\mathbf{k}$ and $\mathbf{l}$. Since $(-\mathbf{u})^2=\mathbf{u}^2$, we see that the square vector turns through the angle 2π, or -2π.



Let us now consider simultaneously the rotating of the unit vector and of its square (Figure). It is evident that during the rotation of the unit vector $\mathbf{u}$ through an angle 2π ($\mathbf{u}^2$ will then turn through an angle $\pm 4\pi$) $\mathbf{u}$ coincides with a vector $\mathbf{f}$ which is parallel to its own square in the same direction: $\mathbf{f}^2=\rho\mathbf{f}$ with $\rho>0$, and with a vector $\mathbf{j}$ whose square lies in the same direction as the vector $-\mathbf{f}$, that is, $\mathbf{j}^2=-\sigma\mathbf{f}$ with $\sigma>0$. Put $\mathbf{e}=(1/\rho)\mathbf{f}$ and $\mathbf{i}=(1/\sqrt{\rho\sigma})\mathbf{j}$, then we obtain

$$\mathbf{e}^2=\frac{1}{\rho^2}\mathbf{f}^2=\frac{1}{\rho^2}\rho\mathbf{f}=\mathbf{e}$$

and

$$\mathbf{i}^2=\frac{1}{\rho\sigma}\mathbf{j}^2=-\frac{1}{\rho\sigma}\sigma\mathbf{f}=-\mathbf{e}.$$

The vectors $\mathbf{e}$ and $\mathbf{i}$ are surely not parallel, since otherwise their squares would have the same direction. Hence it follows that $\mathbf{e}$ and $\mathbf{i}$ span the plane and therefore we may write each vector $\mathbf{a}$ in the plane in the form

$$\mathbf{a}=\xi\mathbf{e}+\eta\mathbf{i}$$

with suitable real numbers ξ and η. We have still to know which vector of the plane is the vector $\mathbf{ei}$.

We proceed to prove that $\mathbf{ei}=\mathbf{i}$, or, more generally, $\mathbf{ea}=\mathbf{a}$ for all vectors $\mathbf{a}$. The equation $\mathbf{e}^2=\mathbf{e}$ implies $\mathbf{e}^2\mathbf{a}=\mathbf{ea}$ or $\mathbf{e}(\mathbf{ea}-\mathbf{a})=0$ for all $\mathbf{a}$. Since there are no divisors of zero, it follows that $\mathbf{ea}-\mathbf{a}=0$, *i.e.* $\mathbf{ea}=\mathbf{a}$, as stated.

From the results obtained so far we may conclude that multiplication of two vectors is to be performed as though $\mathbf{e}$ and $\mathbf{i}$ were equal to the real number 1 and the imaginary unit $\sqrt{-1}$, respectively, *i.e.* we multiply according to the rules of multiplication of complex numbers.

Now use an affine transformation of the plane which carries the vectors $\mathbf{e}$ and $\mathbf{i}$ into two unit vectors perpendicular to each other; such a transformation clearly always exists. Then define the operations in the new vector system as induced by this affine transformation. It is obvious that we arrive just at the vectors of the complex plane with the well-known definitions of operations.

This completes the proof of the theorem.

Remark 1. As the proof above shows, the requirement of associativity can be replaced by commutativity. Also, it is worth while noticing that if we suppose neither associativity nor commutativity, the theorem is no longer valid. It is readily checked that if for two non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$ we put

$$\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{a}, \qquad \mathbf{ab} = -\mathbf{ba} = \mathbf{b}$$

(associativity fails to hold: $\mathbf{b}^2\mathbf{b}\neq\mathbf{bb}^2$), then we obtain an algebra without divisors of zero, but not isomorphic to the complex number system.

Remark 2. In the above proof we have made use of continuity; this can, however, be avoided, as is shown by the following argument.

For an arbitrary vector $\mathbf{u}\neq 0$ the vectors $\mathbf{au}$ and $\mathbf{bu}$ span the plane provided that the same holds for $\mathbf{a}$ and $\mathbf{b}$ (zero divisors do not exist!), therefore for any given vector $\mathbf{v}$ we have

$$\mathbf{v} = \alpha\mathbf{au} + \beta\mathbf{bu} = (\alpha\mathbf{a} + \beta\mathbf{b})\mathbf{u} = \mathbf{xu}$$

with $\mathbf{x}=\alpha\mathbf{a}+\beta\mathbf{b}$ and α, β suitable real numbers. Hence right-division and, similarly, left-division by any non-zero vector exists. We conclude that the vectors not equal to 0 form a group under multiplication with, say, $\mathbf{e}$ as identity. Then each vector has the form $\gamma\mathbf{e}+\delta\mathbf{d}$ with $\mathbf{d}$ an arbitrary vector not parallel to $\mathbf{e}$. Hence the multiplication is commutative.

Now from $\mathbf{d}^2=\lambda\mathbf{e}+\mu\mathbf{d}$ we have $(\mathbf{d}-\frac{1}{2}\mu\mathbf{e})^2=(\lambda+\frac{1}{4}\mu^2)\mathbf{e}$ where the real number $\lambda+\frac{1}{4}\mu^2=\kappa$ must be negative, since $\kappa\geqq 0$ would imply that the vectors $\mathbf{d}-\frac{1}{2}\mu\mathbf{e}\pm\sqrt{\kappa}\mathbf{e}$ are divisors of zero. Consequently, for the vector

$$\mathbf{i} = \frac{\mathbf{d} - \frac{1}{2}\mu\mathbf{e}}{\sqrt{-\kappa}}$$

we get $\mathbf{i}^2=-\mathbf{e}$, *q.e.d.*

THE BERNOULLI DIFFERENTIAL EQUATION

M. R. SPIEGEL, Rensselaer Polytechnic Institute

In the differential equation of Bernoulli

$$\frac{dy}{dx} + Py = Qy^n \tag{1}$$

where P and Q are functions of x, the usual method of solution is to place

$$v = y^{1-n} \tag{2}$$

and solve the resulting linear equation. This linear equation is

$$\frac{dv}{dx} + (1 - n)Pv = (1 - n)Q. \tag{3}$$

Since an integrating factor of (3) is $e^{\int_0^x (1-n)Pdx}$, we may write (3) as

$$\frac{d}{dx}\left[ve^{\int_0^x (1-n)Pdx}\right] = (1 - n)Qe^{\int_0^x (1-n)Pdx} \tag{4}$$

which yields the solution

$$v = \left\{e^{\int_0^x (n-1)Pdx}\right\}\left\{\int_0^x (1 - n)Qe^{\int_0^x (1-n)Pdx}dx + v_0\right\}. \tag{5}$$

Here v_0 is the value assumed by v when $x=0$. If we take y_0 to be the value assumed by y when $x=0$, the solution of (1) is

$$y = \left\{e^{-\int_0^x Pdx}\right\}\left\{y_0^{1-n} + (1 - n)\int_0^x Qe^{\int_0^x (1-n)Pdx}dx\right\}^{1/(1-n)}. \tag{6}$$

When the differential equation (1) is solved in the various textbooks, it is usually pointed out that one assumes n not equal to 0 or 1. The purpose of this article is to show that the solutions obtained for (1), in the supposed exceptional cases where $n=0$ or 1, may actually be obtained from the solution (6).

For the case $n=0$, there is no difficulty. Letting $n=0$ in (6) we obtain

$$y = \left\{e^{-\int_0^x Pdx}\right\}\left\{y_0 + \int_0^x Qe^{\int_0^x Pdx}dx\right\} \tag{7}$$

which is the solution of (1) in the case where $x=0$, $y=y_0$,

For the case $n=1$, there is a difficulty, for (6) then contains an indeterminate factor of the form 1^∞. The limit of (6) as $n\to 1$ may be shown, by the usual methods of dealing with indeterminate forms, to equal

(8) $$y_0 e^{\int_0^x (Q-P)\,dx}$$

which is the solution of (1) with $n=1$ such that $y=y_0$ when $x=0$ as may be easily verified.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1036. *Proposed by W. R. Ransom, Tufts College*

How large an asteroid could a man jump clear off of?

E 1037. *Proposed by A. G. Anderson, Oberlin College*

Show that a necessary and sufficient condition that a real cubic equation $ax^3+bx^2+cx+d=0$ have one real and two pure imaginary roots is that $bc=ad$ and $ac>0$.

E 1038. *Proposed by S. B. Townes, University of Hawaii*

Show that if a number of $3n$ digits is divisible by 27, then any number with the same digits cyclically permuted will also be divisible by 27.

E 1039. *Proposed by I. W. Burr, Purdue University*

Minimize the product of two perpendicular central chords of a given ellipse.

E 1040. *Proposed by H. E. Stelson, Michigan State College*

On page 20 of Rider and Fischer, *Mathematics of Investment*, is the statement, "It does not appear to have been generally recognized that the assumptions underlying the Residuary method are equivalent to those of the Merchant's Rule and hence the two methods yield identical results."

Prove that the Merchant's Rule and the Residuary (Interest at End) method give the same results for the case where the payments are all equal except the last and where the interest payment amounts to more than a single payment.

SOLUTIONS

The Problem of the Three Cowboys

E 1001 [1952, 104]. *Proposed by C. S. Ogilvy, Syracuse University*

Three cowboys who have to watch over cattle in a square pasture divide the square into three equal rectangles by parallel lines in the obvious way. Each then stations himself in the center of one of the rectangles and is responsible only for the cattle in his rectangle.

Show that the square can be divided into three equal areas in another way and a cowboy can be stationed at a point of each, so that (1) the three cowboys have the same "maximum ride," which is the distance from a cowboy's station to the most distant point of his area; (2) this maximum ride is less than it was before; and (3) every point in the square is entrusted to the man whose station is nearest it.

This problem is posed but not solved in Steinhaus, *Mathematical Snapshots* (1950), pp. 34–35.

Solution by Leon Bankoff, Los Angeles, Calif. On one of the parallel lines which trisect the square in the obvious way, two points are located at distances $a/4$ from the sides, where a is a side of the square. Lines through these points directed toward the interior of the square and inclined at 120° to each other meet at a point through which we draw a third line bisecting the larger of the two areas. The area of each of the two trapezoids thus formed is clearly equal to $(a/2)(2a/3)$, or $a^2/3$, so that the three areas are equal.

Cowboy A is stationed in the pentagonal area at the intersection of the two longer diagonals. Cowboys B and C are stationed in the trapezoidal areas at points which are reflections of station A with respect to the boundaries A–B and A–C. Hence stations A, B, C are vertices of an equilateral triangle, the sides of which are bisected perpendicularly by the three boundaries.

All conditions are now fulfilled.

(1) Each cowboy has the same maximum ride (from his station to the outer end of an oblique boundary). This is easily verified by placing the configuration on a rectangular coordinate system.

(2) Since A is now stationed at the center of a diagonal shorter than the original one, his maximum ride, as well as that of the others, is now less than it was before.

(3) Every point in the square is entrusted to the man nearest it, since all points on the common boundaries are equidistant from the stations in the areas separated by them.

Also solved by Julian Braun, W. E. Briggs, F. F. Dorsey, E. I. Gale, Michael Goldberg, J. D. Haggard, J. M. Kingston, L. A. Ringenberg, C. M. Sandwick, Sr., Henry Zatzkis, and the proposer.

The problem has, of course, infinitely many solutions. Kingston and the proposer gave solutions which are symmetrical in a diagonal of the square. It

would be an interesting problem to find the situation which minimizes the maximum ride of the three cowboys.

m-fold Factor of a Polynomial

E 1002 [1952, 104]. *Proposed by T. M. Apostol, California Institute of Technology*

Let A, B, C represent polynomials in a given number-field F, and let A' denote the derivative of A. Suppose B is an m-fold factor of A, *i.e.*, $A=B^mC$ and $B\nmid C$. It is well known that B is at least an $(m-1)$-fold factor of A'. The example $B=(x-1)^3$, $A=(x-1)^3(x^3-1)^2$ shows that it is possible for B to be an m-fold factor of A'. Is it possible for B to be an $(m+1)$-fold factor of A'?

Solution by G. A. Baker, Jr., California Institute of Technology. We know that if a polynomial P is a prime m-fold factor of a polynomial Q, then it is exactly an $(m-1)$-fold factor of Q'. By the prime factorization theorem we have

$$B = p_1^{j_1} p_2^{j_2} \cdots p_r^{j_r}, \qquad A = p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n},$$

where the p_i are distinct prime factors, $r \leqq n$, $mj_i \leqq m_i$ $(1 \leqq i \leqq r)$, and for some k, $m_k < (m+1)j_k$ (otherwise B^{m+1} would be a factor of A). Since p_i is exactly an (m_i-1)-fold factor of A', if B^{m+1} is a factor of A' then for all i, $(1 \leqq i \leqq r)$, $m_i - 1 \geqq (m+1)j_i$, a contradiction. It follows that B cannot be an $(m+1)$-fold factor of A'.

Also solved by R. Z. Vause, Jr., and the proposer.

The above solution assumes that the underlying field F is of characteristic zero. Vause showed that a contrary state of affairs may exist if F is of positive characteristic p. Suppose, for example, F is the modular field having $p=5$, and take

$$B = x - 1, \qquad A = (x-1)^5(2x^3 - x^2 + x).$$

Then $A' = (x-1)^7$.

A Criterion for $f(x, y)=g(x)+h(y)$

E 1003 [1952, 105]. *Proposed by J. G. Wendel, Yale University*

Let $f(x, y)$ be continuous on the unit square, $0 \leqq x, y \leqq 1$. Show that $f(x, y)$ can be written in the form $f(x, y) = g(x) + h(y)$ if and only if

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \{f(x, y) + f(u, v)\}^2 dxdydudv$$

$$= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \{f(x, y) + f(u, v)\}\{f(x, v) + f(u, y)\} dxdydudv.$$

Solution by J. E. Wilkins, Jr., Nuclear Development Associates, Inc. It is obvious that this equation holds when $f(x, y) = g(x) + h(y)$. Conversely, the

equation asserts that Schwarz' inequality is an equality for the functions $f(x, y)+f(u, v)$ and $f(x, v)+f(u, y)$. These functions must then be proportional, and in fact equal, for it is easy to see that either $f(x, y)$ vanishes identically or the factor of proportionality is unity. If we set $u=v=0$ we find that

$$f(x, y) = f(x, 0) + \{f(0, y) - f(0, 0)\} = g(x) + h(y),$$

as desired.

Also solved by Hyman Kaufman and Jerome Shipman (jointly), and the proposer.

Editorial Note. The assumption of the continuity of $f(x, y)$ is used to ensure that $f(x, y)+f(u, v)$ and $f(x, v)+f(u, y)$ are proportional everywhere, and not merely almost everywhere. More stringent restrictions on $f(x, y)$ lead to other criteria for the desired decomposition. For example, for properly restricted $f(x, y)$ a criterion is that $\partial^2 f/\partial x\partial y \equiv 0$. The interest of the criterion of the problem lies in the weakness of the restriction put on $f(x, y)$.

An Approximation for $\sqrt[n]{a}$

E 1004 [1952, 105]. *Proposed by L. R. White, Washington, D. C.*

(1) The approximation $\sqrt[n]{a}=1+(a-1)/n$ for n large and a close to 1 is well known and frequently used. Show that for large n and all a

$$\sqrt[n]{a} = 1 + (\ln a)/n$$

is a good approximation.

(2) Find

$$\lim_{n\to\infty} \{(k - 1 + \sqrt[n]{a})/k\}^n.$$

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. (1) For all $a>0$

$$\sqrt[n]{a} = e^{(\ln a)/n} = 1 + (\ln a)/n + O(1/n^2).$$

(2) Using the preceding result we find

$$\lim_{n\to\infty} \{(k - 1 + \sqrt[n]{a})/k\}^n = \lim_{n\to\infty} \{1 + (\ln a)/kn + O(1/n^2)\}^n$$

$$= e^{(\ln a)/k} = a^{1/k}.$$

Also solved by Alan Berndt, W. E. Briggs, Kenneth Bush, P. L. Chessin, W. W. Gerlinger, Emil Grosswald, Milton Handel, Vern Hoggatt, Hyman Kaufman and Jerome Shipman (jointly), L. A. Ringenberg, W. R. Scott, M. R. Spiegel, and S. C. Venkataraman.

An Application of the Newton-Raphson Method

E 1005 [1952, 105]. *Proposed by D. W. Dubois, University of Oklahoma*

Let $x_0 \neq 0$ and $a > 0$ be two real numbers. Define

$$x_{n+1} = x_n/2 + a/2x_n, \qquad n = 0, 1, 2, \cdots .$$

Find all values of x_0 and a for which the sequence $\{x_n\}$ converges, and find the limits.

I. *Solution by Hyman Kaufman and Jerome Shipman, Laboratory for Electronics, Boston, Mass.* The expression is recognized as the $(n+1)$st iterate in the well known Newton-Raphson method for finding the roots of $y = a - x^2$, *i.e.*, for solving $x^2 = a$. From the theory of the Newton-Raphson method, or from a glance at the graph of $y = a - x^2$, it can be easily shown that for $x_0 > 0$, $\{x_n\}$ converges to $\sqrt{a}$, while for $x_0 < 0$, $\{x_n\}$ converges to $-\sqrt{a}$.

II. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn.* If we plot $y = x/2 + a/2x$ and $y = x$ we get a geometric insight into the convergence process. The curve is a hyperbola with $x = 0$ and $y = x/2$ as the asymptotes. The intersections are $(\sqrt{a}, \sqrt{a})$ and $(-\sqrt{a}, -\sqrt{a})$. Also, the points of intersection are the maximum and minimum points of the hyperbola. From the plot it follows that the sequence converges for all values of $a > 0$ and $x_0 \neq 0$. If $x_0 > 0$, then $x_n \to \sqrt{a}$; if $x_0 < 0$, then $x_n \to -\sqrt{a}$.

Also solved by Kenneth Bush, A. L. Epstein, E. P. Graney, R. E. Greenwood, Emil Grosswald, C. S. Ogilvy, L. A. Ringenberg, and the proposer.

Greenwood called attention to Whittaker and Robinson, *The Calculus of Observations*, pp. 79–81. A late solution by A. E. Livingston was received.

Packaged Radios

E 1006 [1952, 180]. *Proposed by Leo Moser, University of Alberta*

A wholesale jobber received radios from the manufacturer packed in two types of cartons. Each type held a different number of radios. The shipping clerk found that by judiciously selecting the right number of one or both kinds, he could fill almost any size order without opening a carton. Indeed, there were exactly six orders possible which would require a carton to be opened and repackaged.

One day the manufacturer discontinued the smaller carton and sent a new size, containing a different number of radios. The shipping clerk calculated that with the new set-up there were ten different orders which would be impossible to fill without repacking. How many radios were contained in this new carton?

Solution by C. M. Sandwick, Sr., Easton High School, Easton, Pa. If $(m, n)=1$, the number of positive integers which are not of the form $Am+Bn$, where A and B are non-negative integers, is $(m-1)(n-1)/2$. Moreover, the value of each integer which is not of this form is less than $(m-1)(n-1)$.

Let a be the number of radios packed in each small carton and let b be the number in each large carton of the two original types. Let c be the number of radios packed in each carton of the type which is substituted for the smaller cartons.

We have $(a-1)(b-1)/2=6$ and $(c-1)(b-1)/2=10$. Hence $5(a-1)=3(c-1)$, or $5a-3c=2$. Only one solution of this equation yields a corresponding value of b which is a positive integer greater than a. The unique solution is $a=4$, $c=6$, and $b=5$. Thus, each of the new cartons contained six radios.

When using the original cartons, containing 4 and 5 radios, the clerk could not ship 1, 2, 3, 6, 7, or 11 radios without opening a carton. Using cartons containing 5 and 6 radios, he could not ship 1, 2, 3, 4, 7, 8, 9, 13, 14, or 19 radios without opening a carton.

Also solved by Leon Bankoff, C. R. Berndtson and J. E. Wilkins, Jr. (jointly), Julian Braun, F. F. Dorsey, William Douglas, Vern Hoggatt, Ray Jurgensen, J. M. Kingston, M. S. Klamkin, Sidney Kravitz, Azriel Rosenfeld, Charles Salkind, R. Z. Vause, Jr., Lila Peck Walker, LeRoy Warren, and the proposer.

(a, b)=(a+b, [a, b])

E 1007 [1952, 180]. *Proposed by Harvey Berry, University of Kentucky*

Prove that the G.C.D. of two numbers is equal to the G.C.D. of their sum and their L.C.M.

Solution by C. F. Pinzka, Princeton, N. J. If g is the G.C.D. of a and b, we may write $a=a'g$ and $b=b'g$, where a' and b' are mutually prime. Then the sum and L.C.M. of a and b are $(a'+b')g$ and $a'b'g$, respectively. Both have the common factor g, which is also their G.C.D., since $a'+b'$ and $a'b'$ are mutually prime.

Also solved by A. N. Aheart, F. Bagemihl, Alan Berndt, Julian Braun, Herbert Emich, C. V. Fronabarger, J. D. Haggard, Melvin Henriksen, Vern Hoggatt, Ray Jurgensen, J. D. E. Konhauser, George Millman and Bernard Chovitz (jointly), R. G. Paxman, Azriel Rosenfeld, C. M. Sandwick, Sr., J. S. Shipman, R. H. Sprague, W. L. Strother, D. R. Sudborough, R. P. Tapscott, C. W. Trigg, R. Z. Vause, Jr., Lila Peck Walker, Maud Willey, and R. R. Williams, Jr.

This problem is the same as Problem No. 116, *Mathematics Magazine*, Nov. 1951. Simple counter-examples show that the problem does not extend directly to more than two numbers.

A Recurrent Construction Problem

E 1008 [1952, 180]. *Proposed by R. H. Moorman, Tennessee Polytechnic Institute*

Through a given point draw a line cutting the sides of a given angle so as to form a triangle of given perimeter.

(*Editorial Note*: This is the same as problem 3320 [1929, 291], whose solution is herewith reproduced.)

Solution by D. L. Barrick, University of Colorado. Construction: Let D be the given point, EAF the given angle, and $2p$ the given perimeter. On AE take AZ equal to p and on AF take AY equal to p. Construct a circle tangent to AE and AF at Z and Y respectively. Through D construct a line tangent to the minor arc ZY meeting AE and AF at B and C respectively. This tangent is the required line.

Proof: The perimeter of ABC equals $2AZ$ equals $2p$.

Discussion: There is one and only one solution if D is outside of the given angle or its vertical angle, or if D falls on AZ, AY, or minor arc ZY not including points A, Z, or Y. There are two solutions if D falls within the area bounded by AZ, AY, and minor arc ZY. Otherwise there is no solution. It is assumed here that the sides of the triangle are not to lie upon the prolongations of EA and FA.

Also solved by I. D. Abella, Alan Berndt, C. W. Bruce, F. F. Dorsey, C. V. Fronabarger, J. D. Haggard, John Jones, Jr., M. S. Klamkin, J. D. E. Konhauser, Sam Kravitz, B. R. Leeds, B. Martin, C. F. Pinzka, Charles Salkind, C. M. Sandwick, Sr., A. Sisk, R. H. Sprague, and C. W. Thompson.

N. A. Court pointed out that this problem was discussed at great length by S. A. J. Lhuilier (1750–1840), *Éléments d'analyse géométrique et d'analyse algébrique, appliquées a la recherche des lieux géométriques*, p. 219, ex. 5, Paris and Geneva, 1809. It was also taken up by Eugene Catalan (1814–1894) in his *Théorèmes et problèmes de géométrie élémentaire*, p. 68, prob. 5, 6th ed., Paris, 1879. It has since appeared repeatedly in the problem departments of various periodicals, and is stated in several books on college geometry, such as Court, *College Geometry*, ex. 3, p. 80; Davis, *Modern College Geometry*, ex. 4, p. 38; Taylor and Bartoo, *Introduction to College Geometry*, ex. 2, p. 43.

The following two allied problems, solutions of which are easily suggested by that of the given problem, were proposed by Konhauser and Abella, respectively.

(1) Through a point between the sides of an angle draw a line cutting the sides of the angle so as to form a triangle of minimum perimeter.

(2) Through a point draw a line cutting off from a given triangle a quadrilateral of given perimeter.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4508. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

Find a polynomial $F(x)$ of lowest degree such that $F(x)+\omega^s$ is divisible by $(x-\omega^s)^r$ for $s=0, 1, 2, \cdots, p-1$, where ω is a primitive pth root of unity and r is a given positive integer. (This generalizes a problem in Goursat-Hedrick, *A Course in Mathematical Analysis*, v. 1, p. 32.)

4509. *Proposed by D. J. Newman, Harvard University*

Prove that the most general multiplicative function $f(n)$ such that $f(n+1) \sim f(n)$ is $f(n)=n^a$ for some constant a.

4510. *Proposed by H. P. Thielman, Iowa State College*

If $a_1, a_2, a_3, \cdots$ is a sequence of real numbers such that $a_{n+1}=1/n^p a_n$, where $p>0$, and $n=1, 2, 3, \cdots$, prove that the sequence a_n is monotone non-increasing if and only if $a_1=(\pi/2)^{p/2}$.

4511. *Proposed by W. V. Parker, Alabama Polytechnic Institute*

Solve the Diophantine equation

$$x^2 + y^2 + (xy)^2 = z^2.$$

4512. *Proposed by E. P. Starke, Rutgers University*

Prove that every positive rational number with odd denominator is a sum of a finite number of distinct terms from the sequence $1/3, 1/5, 1/7, \cdots$.

SOLUTIONS

Integers with Large Prime Divisors

4436 [1951, 266]. *Proposed by D. J. Newman, Harvard University*

What is the probability that an arbitrary integer have a prime divisor which is larger than its square root?

I. *Solution by J. D. Swift, University of California, Los Angeles.* The probability is log 2. In the proof of this result we shall make use of Tchebychef's theorem: $\Pi(x)=O(x/\log x)$, and Merten's theorem: $\sum_{p\leqq x}1/p=\log\log x+C+o(1)$. The letters p and q will be reserved for primes.

Let $A(n)$ be the number of positive integers a such that $a\leqq n$, and $p\mid a$ for some $p>\sqrt{a}$. Now the integers a are those multiples kp of primes less than or equal to n such that $k\leqq p-1$ and $kp\leqq n$. Therefore

$$A(n)=\sum_{p\leqq n}\min(p-1,[n/p]).$$

It is clear that no duplications occur in this method of counting; *e.g.* a product pq is counted as a multiple of q if $q>p$ and not as a multiple of p. Since $p-1$ will be the minimum up to $\sqrt{n}$, we have

$$\begin{aligned}A(n)&=\sum_{p\leqq\sqrt{n}}(p-1)+\sum_{\sqrt{n}<p\leqq n}[n/p]\\&=O(\sqrt{n}\,\pi(\sqrt{n}))+n\sum_{\sqrt{n}<p\leqq n}1/p+O(\pi(n)-\pi(\sqrt{n}))\\&=n(\log\log n-\log\log\sqrt{n}+o(1))+O(n/\log n)\\&=n\log 2+o(n).\end{aligned}$$

Therefore $\lim A(n)/n=\log 2$.

II. *Solution by John Todd, National Bureau of Standards.* S. D. Chowla and John Todd have shown that the density of integers n, of which the largest prime factor is greater than any fixed multiple of $n^{1/2}$ is $\log 2=.6931$. See *Canadian Journal of Mathematics*, v. 1, pp. 297–299.

Also solved by P. T. Bateman, J. B. Kelly, E. H. Umberger, and the Proposer.

Evaluation of a Hypergeometric Function

4438 [1951, 343]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Show that the polynomial of degree $2n$ defined by the hypergeometric function $F(-2n,\frac{1}{2};n+1;x)$ takes on the value 1 when $x=4$.

Solution by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y. We make use of a well-known integral representation for the hypergeometric function (Whittaker and Watson, *Modern Analysis*, American Edition, p. 293) to obtain

$$F(-2n,\tfrac{1}{2};n+1;4)=\frac{\Gamma(n+1)}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_0^1 u^{-1/2}(1-u)^{n-1/2}(1-4u)^{2n}du.$$

It is clear that this reduces to 1 in view of the following:

$$\int_0^1 u^{-1/2}(1-u)^{n-1/2}(1-4u)^{2n}du = 2\int_0^{\pi/2} \cos^{2n}\theta(1-4\sin^2\theta)^{2n}d\theta$$

$$= 2\int_0^{\pi/2} (\cos 3\theta)^{2n}d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2n}\phi d\phi = \frac{\Gamma(n+\frac{1}{2})\sqrt{\pi}}{\Gamma(n+1)}.$$

Also solved by F. G. Fender, Anne C. Moldauer, O. E. Stanaitis, Chih-yi Wang, and the Proposer.

Triangle *ABC* and a Locus Problem

4439 [1951, 343]. *Proposed by Josef Langr, Prague, Czecho-Slovakia*

A line m cuts the sides BC, CA, AB of a triangle ABC at L, M, N. Show that the midpoints of the segments AL, BM, CN are on a line n, and find the locus of the point of intersection of m and n as m moves parallel to itself.

Solution by C. S. Ogilvy, Syracuse University. Locate the triangle on oblique coördinate axes with A at the origin, B at the point $(b, 0)$ and C at $(0, c)$. Let L, M, N have the coördinates (p, q), $(0, r)$ and $(-s, 0)$ respectively. Solving for the intersection of m and BC gives

$$p = \frac{bs(c-r)}{cs+br}, \qquad q = \frac{rc(s+b)}{cs+br}.$$

Using these values it is easy to verify that the three midpoints $(p/2, q/2)$, $(b/2, r/2)$ and $(-s/2, c/2)$ are collinear, the slope of n being $(r-c)/(b+s)$.

If μ is the slope of m, then $y=\mu x+r$ is the equation of m. The equation of n is $(d-2y)/(a-2x)=(r-c)/(b+s)$. Using the fact that $\mu=r/s$, we eliminate r and s between the equations of m and n, finding the locus of their intersections to be the parabola

$$\mu x^2 - 2xy + (1/\mu)y^2 + 2cx + 2by - bc = 0,$$

whose axis is parallel to m.

Also solved by Emil Grosswald, R. F. Hays, M. S. Klamkin, George Millman (first part), W. O. Pennell, O. J. Ramler, L. A. Ringenberg, Azriel Rosenfeld (first part), A. Sisk, F. Underwood, and the Proposer.

Editorial Note. Ramler notes that n is the Newton line of the quadrilateral composed of the sides of triangle ABC together with m. He also shows that the parabola is inscribed in the medial triangle of ABC, a fact that is easily verified analytically in the above solution.

In generalizing the problem, Grosswald allows m to vary so as always to pass through a fixed point P. The locus of intersections of m and n is then in general a curve of the third degree having three real asymptotes parallel to the three

sides of ABC. If, however, P is on a side of ABC, then all lines n also pass through a fixed point Q, and the locus of intersections is a hyperbola passing through P and Q and having asymptotes parallel to the other two sides of ABC.

A Summation

4440 [1951, 343]. *Proposed by R. G. Stoneham, University of California, Berkeley.*

Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}B_{2k}\pi^{2k}}{(2k+1)!}$$

where B_{2k} are the even Bernoulli numbers.

Solution by O. E. Stanaitis, St. Olaf College, Northfield, Minnesota. From the well-known formula

$$x \cot x = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k}B_{2k}}{(2k)!} x^{2k},$$

by integrating between the limits 0 and $\pi/2$ we have

$$\int_0^{\pi/2} x \cot x dx = \pi/2 + \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{(-1)^k B_{2k}\pi^{2k}}{(2k+1)!}.$$

Integration by parts gives

$$\int_0^{\pi/2} x \cot x dx = -\int_0^{\pi/2} \log \sin x dx = \frac{\pi}{2} \log 2,$$

a familiar definite integral. Hence the value of the proposed sum is $1-\log 2$.

Also solved by Hyman Kaufman, M. S. Klamkin, W. Seidel, Chih-yi Wang, and the Proposer.

Polynomials with Positive Coefficients

4441 [1951, 343]. *Proposed by R. M. Cohn, Rutgers University*

If a polynomial equation $f(x)=0$, with integral coefficients has no positive roots, there exists a polynomial $g(x)$ with integral coefficients such that all coefficients of $f(x)\cdot g(x)$ are positive.

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. It will suffice to prove the theorem for $q(x)=x^2-ax+b$, $(a, b>0, a^2<4b)$. Let $p(x)=(1+x)^k$. Then all the coefficients of $q(x)\cdot p(x)$ will be positive if

$$\binom{k}{r} - a\binom{k}{r-1} + b\binom{k}{r-2} > 0 \tag{1}$$

for $r=1, 2, \cdots, k+1$. This is always possible for k sufficiently large.

$f(x)$ can be factored into linear factors like $x+c$, $c \geqq 0$, and quadratic factors like $q(x)$. We may then take $g(x)$ to be $(1+x)^K$ where K is the sum of the k's determined as above for the individual quadratic factors of $f(x)$. Since the coefficients of $f(x)$ are integers, the coefficients of $f(x) \cdot g(x)$ will be positive integers.

Also solved by Emil Grosswald, L. J. Paige, George Piranian, O. E. Stanaitis, M. A. Kropnitsky and the Proposer.

Editorial Note. The existence of suitable k may be seen as follows. (1) reduces to

$$(k - r + 2)(k - r + 1) - ar(k - r + 2) + br(r - 1) > 0$$

which is true if

$$2k > (2 + a)r - 3 + \sqrt{1 + (2a + 4b)r - (4b - a^2)r^2} \tag{2}$$

and the radical is real; if the radical is imaginary (1) is true for all k. Evidently the right member of (2) has a maximum value and we need only choose k greater than this maximum.

Paige and Piranian refer to a proof by E. Meissner (*Math. Annalen*, v. 70 (1911), p. 223), and to the following generalization by D. R. Curtiss (*Math. Annalen*, v. 73 (1913) p. 424): *If $f(x)$ is any polynomial with all its coefficients real, there exist polynomials $f_1(x)$ such that when the product $f_2(x) = f_1(x) \cdot f(x)$ is arranged according to ascending or descending powers of x, the number of variations of sign presented by its coefficients is exactly equal to the number of positive roots of $f(x)$.* See also Pólya and Szegö, *Aufgaben und Lehrsatze*, v. II, problem 190, section 5.

Zeros of a Complex Polynomial

4442 [1951, 343]. *Proposed by Michael Golomb, Purdue University*

If a polynomial (not a constant) with complex coefficients is of the same modulus at two points z_1, z_2 of the z-plane, show that it has at least one zero in the open half-plane

$$R\{\bar{z}(z_1 - z_2)(|z_1| - |z_2|)\} > 0$$

if $|z_1| \neq |z_2|$, and at least one zero in each of the closed half-planes

$$R\{\bar{z}(z_1 - z_2)\} \geqq 0, \qquad R\{\bar{z}(z_1 - z_2)\} \leqq 0$$

if $|z_1| = |z_2|$.

Solution by S. H. Gould, Purdue University. Let the coefficient of the highest power of z be unity. If w_j are the zeros of $f(z)$, we may write

$$\prod |z_1 - w_j| = |f(z_1)| = |f(z_2)| = \prod |z_2 - w_j|,$$

so that at least one w_j lies in the half-plane $|z_2 - z| \geqq |z_1 - z|$, since otherwise we would have $\prod |z_1 - w_j| > \prod |z_2 - w_j|$.

If $z_j = x_j + iy_j$, the inequality $|z_2 - z| \geqq |z_1 - z|$ becomes

$$(x_2 - x)^2 + (y_2 - y)^2 \geqq (x_1 - x)^2 + (y_1 - y)^2,$$
$$2x(x_1 - x_2) + 2y(y_1 - y_2) \geqq x_1^2 + y_1^2 - x_2^2 - y_2^2,$$
$$2R\{\bar{z}(z_1 - z_2)\} \geqq |z_1|^2 - |z_2|^2,$$

so that

$$R\{\bar{z}(z_1 - z_2)(|z_1| - |z_2|)\} > 0 \qquad \text{if } |z_1| > |z_2|,$$
$$R\{\bar{z}(z_1 - z_2)\} \geqq 0 \qquad \text{if } |z_1| = |z_2|.$$

The rest of the problem results from interchanging z_1 and z_2.

Also solved by Paul Cohen and the Proposer.

Real Zeros of Polynomials

4443 [1951, 422]. *Proposed by R. H. Breusch, Amherst College, Massachusetts.*

If

$$P_n(z) \equiv \sum_{p=0}^{[n/2]} (-1)^p \frac{z^{n-2p}}{(n-2p)!} = \frac{z^n}{n!} - \frac{z^{n-2}}{(n-2)!} + \cdots,$$

prove that the number of real zeros of $P_n(z)$ is

$$N_n = \frac{2n}{e\pi} + \frac{\log n}{e\pi} + O(1).$$

Solution by the Proposer. **1.** Let

$$R_n(x) \equiv \sum_{p=0}^{\infty} (-1)^p \frac{x^{n+2p}}{(n+2p)!} = \frac{x^n}{n!} - \frac{x^{n+2}}{(n+2)!} + \cdots.$$

Since

(a) $R_2(x) = 1 - \cos x \geqq 0$

(b) $R_n'(x) = R_{n-1}(x)$ for $n \geqq 1$

(c) $R_n(0) = 0$ fos $n \geqq 1$,

it follows by complete induction that $R_n(x)$, $n \geqq 3$, is positive and increasing for positive values of x.

2. Call a_n the (one and only) positive value such that $R_n(a_n) = 1$. Now

$$R_{n+2}(x) = P_n(x) \pm \begin{cases} \sin x \\ \cos x \end{cases}.$$

Therefore for $x > a_{n+2}$, $P_n(x) \neq 0$, since

$$|P_n(x)| > 1 - \begin{cases} |\sin x| \\ |\cos x| \end{cases}.$$

For $0<x<x_{n+2}$, the graph of $P_n(x)$ will intersect the x-axis as many times (or at the utmost, once more or once less) as the $\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}$ curve, that is $\frac{a_{n+2}}{\pi}+O(1)$ times. And since $P_n(x)$, as an even or an odd function, has its zeros symmetric with respect to 0,

$$N_n = 2a_{n+2}/\pi + O(1).$$

3. Let Q_n be given by

$$R_n(x) = \frac{x^n}{n!}\left(1 - \frac{x^2}{(n+1)(n+2)} + \cdots\right) = \frac{x^n}{n!}Q_n(x).$$

Since

$$2n^{n+1/2}e^{-n} < n! < 3n^{n+1/2}e^{-n},$$

$$R_n\left(\frac{n}{e}\right) < \frac{n^n}{e^n n!} < 1.$$

Now

$$R_n\left(\frac{n}{2}\right) > \frac{n^n}{2^n n!}\left(1 - \frac{1}{4}\right) > \frac{(e/2)^n}{4\sqrt{n}} > 1 \qquad \text{for } n \geqq 9.$$

Thus $n/e<a_n<n/2$, and $1-\frac{1}{4}<Q_n(a_n)<1$. The rest of the proof proceeds as follows:

$$\frac{3}{4}\frac{a_n^n}{n!} < R_n(a_n) = 1 < \frac{a_n^n}{n!},$$

$$2(n/e)^n\sqrt{n} < n! < a_n^n < 4n!/3 < 4(n/e)^n\sqrt{n},$$

$$\frac{n}{e}e^{\log n/2n+\log 2/n} < a_n < \frac{n}{e}e^{\log n/2n+\log 4/n},$$

$$a_n = \frac{n}{e}\left(1 + \frac{\log n}{2n} + O\left(\frac{1}{n}\right)\right) = \frac{n}{e} + \frac{\log n}{2e} + O(1),$$

$$a_{n+2} = \frac{n+2}{e} + \frac{\log (n+2)}{2e} + O(1) = \frac{n}{e} + \frac{\log n}{2e} + O(1),$$

and finally,

$$N_n = \frac{2n}{e\pi} + \frac{\log n}{e\pi} + O(1).$$

More detailed care with the last steps will yield the result

$$N_n = \frac{2n}{e\pi} + \frac{\log n}{e\pi} + k_n, \qquad |k_n| < 4.$$

Isosceles Tetrahedron, Hyperbolic Sets of Lines

4446 [1951, 702]. Corrected. *Proposed by Victor Thébault, Tennie Sarthe, France*

In an equifacial tetrahedron the lines which join the vertices to the circumcenters (or the orthocenters) of the opposite faces form a hyperbolic system. If the lines are concurrent, the tetrahedron is regular and conversely.

Solution by the Proposer and W. E. Byrne, Virginia Military Institute. Lines joining the midpoints of opposite edges of an equifacial tetrahedron pass through the centroid G, are mutually perpendicular and are axes of symmetry of the tetrahedron. Using them as axes of a rectangular coordinate system, the vertices may be taken as

$$A(a, -b, -c) \qquad B(-a, b, -c)$$
$$C(-a, -b, c) \qquad D(a, b, c)$$

with a, b, c positive. The tetrahedron $ABCD$ is regular if and only if $a=b=c$. The faces ABC, DCB, CAD, BAD are congruent and are symmetric two by two with respect to the axes.

If a line δ is drawn from vertex D to a point D' in face ABC, the lines α, β, γ, symmetric to δ with respect to the axes, pass through the other vertices and meet the opposite faces in points A', B', C' homologous to D' in face ABC. If δ has the equations

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = 0 \qquad lmn \neq 0,$$

then α, β, γ are given by

$$\frac{x-a}{l} = \frac{y+b}{-m} = \frac{z+c}{-n}, \qquad \frac{x-a}{-l} = \frac{y-b}{m} = \frac{z+c}{-n},$$
$$\frac{x+a}{-l} = \frac{y+b}{-m} = \frac{z-c}{n},$$

respectively. If δ does not cut the axes Gx, Gy, Gz and is not parallel to face ABC, then the four lines $\delta, \alpha, \beta, \gamma$ are hyperbolic, *i.e.*, it is easily established that there is a line through D which meets α, β, γ; likewise a line through A which meets β, γ, δ; etc. Evidently, if A', B', C', D' are circumcenters (or orthocenters) we have a special case.

If the tetrahedron is regular (all faces equilateral triangles) the circumcenters

of the faces are also centroids of the faces. Thus, if δ is taken as the line joining D to the centroid of face ABC; $\delta, \alpha, \beta, \gamma$ will meet at G.

Lower Bound for the Zeros of an Analytic Function

4447 [1951, 423]. *Proposed by D. J. Newman, Harvard University*

Let $f(z)$ be analytic in $|z| \leqq 1$. Suppose $|f(z)| \leqq 2$ there and $f(0) = 1$. Show that $f(z)$ has no zeros for $|z| \leqq \frac{1}{2}$.

Solution by Peter Ungar, University College, London, England. Let $f(a) = 0$.

$$g(z) = \frac{\bar{a}z - 1}{z - a} f(z)$$

is regular in $|z| < 1$, and $|g(z)| \leqq 2$ on $|z| = 1$. By the maximum modulus theorem

$$2 \geqq |g(0)| = \frac{1}{|a|} |f(0)| = \frac{1}{|a|},$$

whence $|a| \geqq \frac{1}{2}$. Equality can occur only in case $g(z) \equiv g(0) = 1/a$, so that

$$f(z) = \frac{z - a}{a(\bar{a}z - 1)}.$$

Also solved by P. T. Bateman, R. P. Boas, Jr., R. H. Brown, Paul Cohen, B. Kelly, M. S. Klamkin, A. J. Lohwater, C. S. Ogilvy, M. S. Robertson, W. Seidel, O. E. Stanaitis, J. L. Ullman, H. S. Wall, and the Proposer.

Editorial Note. Most of the solvers note that the equality sign in $|z| \leqq \frac{1}{2}$ of the original statement should be omitted. Cohen and Robertson remark that the original statement remains true if 2 and $\frac{1}{2}$ are replaced by M and $1/M$, $M > 1$. Jensen's theorem (Titchmarsh, *Theory of Functions*, Oxford, 1932, 3.61) and Schwartz's lemma (Titchmarsh, *loc. cit.*, 5.2) are employed in several solutions. Bateman and Stanaitis refer to previous proofs by Landau (*Bull. Soc. Math. France*, v. 33, 1905, pp. 251–261; *Tôhoku Math. Journ.*, v. 5, 1914, pp. 97–116).

A Diophantine Equation

4448 [1951, 495]. *Proposed by Jekuthiel Ginsburg, Yeshiva College, New York City*

Solve in positive integers

$$z^4 = \frac{ax^2 + by^2}{a + b},$$

where a and b are given integers.

I. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York.* The given equation will be satisfied if

$$an(z^2 + x) = m(y - z^2), \qquad m(z^2 - x) = bn(y + z^2),$$

m and n being arbitrary integers. Therefore

$$x:y:z^2 = |\, m^2 - 2bmn - abn^2 \,| : |\, m^2 + 2amn - abn^2 \,| : (m^2 + abn^2),$$

is a solution provided the constant of proportionality, k, is so chosen that $k(m^2+abn^2)$ is a perfect square.

Other solutions may be obtained by solving $z^2=m^2+abn^2$. See L. E. Dickson, *History of the Theory of Numbers*, v. 2, p. 425. One form of the solution is

$$m:n:z = |\, p^2 - abq^2 \,| : 2pq : (p^2 + abq^2).$$

II. *Solution by P. Somanatham and K. Subbarao, M. R. College, Vizianagram, South India.* Set $x=p-bq$, $y=p+aq$. Then

$$\frac{ax^2 + by^2}{a + b} = p^2 + abq^2.$$

Now numbers of the form p^2+abq^2 are multiplicative, that is, the product of two is another of the same form. The formula is

$$(1) \qquad (u^2 + abv^2)(m^2 + abn^2) = (mu \pm abnv)^2 + ab(mv \pm nu)^2.$$

Hence, if $z=p_1^2+abq_1^2$, (1) implies

$$z^2 = p_2^2 + abq_2^2, \qquad p_2 = |\, p_1^2 - abq_1^2 \,|, \qquad q_2 = 2p_1q_1.$$

Then, by (1) again,

$$z^3 = p_3^2 + abq_3^2, \qquad p_3 = |\, p_1p_2 \pm abq_1q_2 \,|, \qquad q_3 = |\, p_1q_2 \mp p_2q_1 \,|.$$

By repeated use of (1) as a recurrence relation we have

$$z^s = p_s^2 \pm abq_s^2, \qquad p_s = |\, p_1p_{s-1} \pm abq_1q_{s-1} \,|, \qquad q_s = |\, p_1q_{s-1} \mp p_{s-1}q_1 \,|,$$

so that infinitely many solutions of $z^s=p^2+abq^2$ and hence of

$$z^s = \frac{ax^2 + by^2}{a + b}$$

may be obtained for every positive integer s.

In particular for $s=4$, if the above substitutions are carried out, a solution of the proposed problem is obtained:

$$x = |\, p_1^4 - 4bp_1^3q_1 - 6abp_1^2q_1^2 + 4ab^2p_1q_1^3 + a^2b^2q_1^4 \,|,$$
$$y = |\, p_1^4 + 4ap_1^3q_1 - 6abp_1^2q_1^2 - 4a^2bp_1q_1^3 + a^2b^2q_1^4 \,|,$$
$$z = p_1^2 + abq_1^2.$$

Aslo solved by E. Morgantini, R. Venkatachalam Iyer, and the Proposer.

Series Conditionally Convergent Everywhere on the Unit Circle

4449 [1951, 495]. *Proposed by D. J. Newman, Harvard University*

The series $\sum z^n/n$ converges everywhere on the unit circle except at $z=+1$. Can $\pm$ signs be introduced so that $\sum \pm z^n/n$ will converge everywhere on the unit circle with no exceptions?

Solution by Frederick Bagemihl, University of Rochester, New York. The answer is affirmative. The series of Pringsheim,

$$\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{n}]}}{n} z^n,$$

as is well known, converges everywhere on the unit circle with no exceptions. (*Münch. Berichte* XXX, 1900; quoted also in Knopp, *Problem Book in the Theory of Functions*, vol. I, New York, 1948.)

The series of problem 4384 [1951, 573] is very similar and possesses the same property.

Also solved by Joshua Barlaz, M. S. Klamkin, V. Linis, David Mandelbaum, O. E. Stanaitis, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Linear Transformations in n-Dimensional Vector Space. By H. L. Hamburger and M. E. Grimshaw. Cambridge University Press, 1952. 10+195 pages. $4.50.

This book was written to "introduce the ideas and methods of the theory of linear transformations in Hilbert space by using them to present the elements of the theory in a finite dimensional vector space."

In the first chapter, the vectors in a finite dimensional space are introduced in their concrete realization as n-tuples of complex numbers. The elements of the unitary structure and geometry of finite dimensional spaces are studied. Theorems about linear manifolds are proved, and applications are made to the solution of linear equations. The elementary algebra of linear transformations concludes the chapter.

The preliminary spectral properties of such special linear transformations as the hermitian, normal, unitary transformations, and projectors are established in the second chapter.

Chapter III contains a proof of the spectral theorem for hermitian transformations H by Hilbert's method (characteristic values are obtained by maximizing the form (Hx, x) on the unit sphere). Eigenvalue inequalities are established, and the functional calculus for hermitian transformations is developed. Commutativity properties of hermitian transformations are studied, and the chapter concludes with a development of the foregoing for normal transformations.

Chapter IV deals with the general linear transformation on n-dimensional space. Various spectral properties are established. A functional calculus for regular functions of linear transformations is developed. The Jordan canonical form is derived first for nilpotent and then for arbitrary linear transformations. Commutativity properties of general linear transformations are studied.

In the fifth and final chapter, finite-dimensional unitary spaces are now introduced abstractly for the purpose of analyzing the pencil $H-\lambda G$, with H hermitian and G positive. Conditions are stated for linear transformations to be hermitian or normal in some scalar product. Applications are made to the dynamical theory of small oscillations.

This book has a certain amount to recommend it. There is a wealth of material to be found in it which is usually not contained in an elementary text. The eigenvalue inequalities, the physical applications, and the analytical techniques are but a few examples of this. Some of the more powerful methods of modern functional analysis are illustrated in the simplified finite dimensional situation in such a manner as to make them readily accessible to the beginning student.

On the other hand, the reviewer found many objectionable features to this book. The notation is made clumsy by using superscripts on letters to denote vectors in a set of vectors and subscripts on letters for complex numbers in a set. The non-invariant concept of "real vector" in unitary space barely merits the attention the authors pay it. The authors leave the reader with the impression that separability is an inherent rather than an assumed (and somewhat inessential) property of Hilbert space (cf. p. 5). At several points, the authors actually go counter to their purpose of developing the finite dimensional theory in a manner most closely adaptable to Hilbert space, and nothing is saved by their procedure: on page 31, they define a regular transformation as one for which the inverse exists and a singular transformation as one of rank less than n; on page 36, they establish the associative law for multiplication of linear transformations by writing down the general element of the matrix obtained by each method of bracketing; on page 42, the spectrum of a linear transformation is defined as the set of all eigenvalues, where an eigenvalue is a number in the point spectrum of the transformation—this is not the spectrum for a transformation on Hilbert space and is unnecessarily misleading in view of the simple possibility of defining the spectral values as those numbers λ for which $A-\lambda I$ is singular. The concept of rank is used freely, but determinants are studiously avoided "because determinants can be used in Hilbert space only in very special cases." A proof of the spectral theorem is adopted which the authors point out is

adaptable only to completely continuous hermitian transformations in Hilbert space. Modern developments have made available simpler, more illuminating, and more elegant proofs of the spectral theorem which yield at the same time the simultaneous spectral decomposition of any family of commuting normal transformations. It should be remarked, however, that the proof given leads naturally to the discussion of the eigenvalue inequalities. The important polar decomposition of a linear transformation into the product of a partially isometric (unitary in the finite case) transformation and a positive one is not discussed. Our final objection is to the lack of motivation presented for the development of the Jordan canonical form.

R. V. KADISON
Institute for Advanced Study

The Design and Analysis of Experiments. By Oscar Kempthorne. New York, John Wiley & Sons, Inc. London, Chapman & Hall, Limited, 1952. 19+631 pages. $8.50.

The aim of this book, according to the Preface, is "to give a description of the design of experiments from as broad a view as possible and to relate the subject matter of this field of statistics to the general field of statistics and to the general problem of experimental inference." This aim the author has carried out well. He gives a systematic treatment of the mathematical theory upon which designs of experiments are based, enabling the research worker to make more intelligent use of these designs, both in planning them and analyzing them.

The formulation and testing of hypotheses are the important features of scientific method in general and of statistical method in particular. A hypothesis can be verified, although not proved, by experiments. That is, the results of the experiment may be in accord with the hypothesis, but they might equally well be in accord with some other hypothesis. On the other hand, a hypothesis can be disproved by the results of the experiment. Now the testing of a hypothesis requires the collection of observations and it is the function of experimental design to provide a pattern of the observations to be collected so that they will be relevant to the hypothesis to be tested and so that they will be accumulated as economically as possible, from the standpoint of time, labor, money, *etc.*

The earlier chapters of the book discuss the principles of statistical designs, elementary statistical notions, the theory of least squares, *etc.* Then come chapters on such topics as multiple regression and the analysis of variance, multiple classifications, randomization and randomized experiments (including randomized blocks and Latin squares), plot technique, factorial experiments, confounding, split-plot experiments, fractional replication, and lattice designs.

Although the treatment is very mathematical in character, a considerable amount of descriptive matter, including illustrative examples, is given.

Before concluding the review of this book it may not be inappropriate to

state its relation to two other recent books on the same subject, Cochran and Cox, *Experimental Designs*, and Mann, *Analysis and Design of Experiments*. It complements the former, which has to do with how an experiment is actually designed, in that it explains the underlying reason for designing it in a particular way. It is much more comprehensive and contains much more explanatory matter than Mann's brief treatise.

The author has in mind three groups of users: "the experimenter, who needs to understand the theory and practice of the statistical designs; the consulting statistician, who wants a description of basic theory and methodology of designs; and the mathematical statistician, who is concerned with the validity of the application of statistical techniques to the analysis of data." A member of any one of these groups would be rewarded by a diligent study of the book or of those parts in which he has a special interest.

P. R. RIDER
Washington University

Statistical Methodology Reviews, 1941–1950. Edited by Oscar Krisen Buros. John Wiley and Sons, Inc., New York: Chapman and Hall, Ltd., London, 1951. Pp. x+457. Price $7.00.

In attempting to write a review of Professor Buros' book, one cannot help having a curious feeling. Various authors wrote books on statistical methodology. Then each of these books was reviewed in several journals. Then Professor Buros collected the reviews, edited them and published his book. Now, I am writing a review of Professor Buros' review of the various reviews. If all goes well, somebody may publish a review of my review of . . . , *etc.* All this sounds like a malicious joke on my part. Actually, of course, the situation calls for a joke but the present one is not intended to be malicious. On the contrary in my opinion, both Dr. Buros' aims and his performance deserve deepest respect.

As stated in the Preface, "This volume is the third in a series of publications prepared to assist research workers and statisticians to locate and evaluate the statistical methodology books in all fields in which statistical methods are used." This primary objective is certainly achieved by Professor Buros and whoever wishes to learn several opinions on a given statistical book published in the decade, 1941–50, will find them collected or, at least, abstracted in the present volume. However, the collection of this kind of information is not the only objective of Professor Buros. In addition to this he has two more objectives which, to the present writer's mind, are more important than the first. One of them is to improve the quality of the reviews (i) "by stimulating editors to take greater pains to choose competent reviewers who have the industry and the courage to contribute frankly critical reviews after a careful study of the book" and (ii) "by stimulating reviewers 'to take their responsibility more seriously' by refusing to review books that they cannot, or will not, appraise

competently and honestly." Finally, the last objective of Dr. Buros is "to discourage the writing and publication of stereotyped textbooks written by persons ignorant of modern developments in statistical theory."

It is difficult to predict with certainty whether or not the last two mentioned aims of Dr. Buros will be attained. Much depends on how widely his book will be seen by persons concerned. All one can say is that a number of reviewers will have good reason to blush when reading their own writings printed side by side with those by other authors all concerned with the same book. One could also risk a similar assertion regarding some of the authors of books. The only question is whether or not these persons will look at the collection of Professor Buros or will try to keep away from it.

There is still another useful purpose that Professor Buros' book can serve although he does not mention it directly. The present period in the life of universities in this country and also in several countries abroad is characterized by a drive towards statistics. Departments of statistics or, at least statistical cells within other departments such as mathematics, are yearly established in more and more institutions of higher learning. In each such instance somebody takes the initiative and usually this somebody is not a professional statistician. His first problem is, then, to find and to bring in a person who could start and develop a suitable program of instruction and research in statistics. Naturally, the choice of this person is most important because, given the opportunity, he will establish a tradition which would be very difficult to eradicate. In situations of this kind, the chairman of the mathematics or of the economics or of the agriculture department, whoever is in charge of finding a suitable statistician, will find it profitable to consult Professor Buros' book. For his prospective candidate may happen to be the author of a book which he should not have written. Alternatively this candidate may be one of the reviewers "who cannot or will not" appraise other authors' works competently and honestly. Judgments in such matters are frequently difficult and a confrontation of several opinions on the same subject is likely to be very useful.

J. NEYMAN
University of California

Introduction to Mathematical Thinking. By Friedrich Waissman. With a foreword by Karl Menger. Translated by T. J. Benac. Frederick Ungar Publishing Co., New York.

The German text of Dr. Waissman's book, subtitled "The Formation of Concepts in Modern Mathematics," appeared first in 1936. Like several other books before and since its time, its aim was to give the non-mathematician a taste of the special flavor of mathematics, and to provide him with an insight into the nature of mathematical activity. Unlike some other books, the aim of this book —as Professor Menger says in his foreword to it—was realized, even though there may be as many disagreements as agreements with Dr. Waissman's discussion of, say, the principle of mathematical induction.

The aim of any translation is obvious. It is also obvious that the prime requirements of a good translation are: consistently good grammar and syntax, reasonably familiar words, and a precise transmission of the original meaning. There are too many sentences in the present translation which do not meet these requirements. When one also observes a somewhat careless editing and printing, one is driven to say that the present translation is a disservice to a fine book.

BEN DUSHNIK
University of Michigan

NEW BOOKS RECEIVED

La Geometrie Descriptive et ses Applications. By Andre Delachet and Jean Moreau. Paris, Presses Universitaire de France, 1952. 120 pages. 150 francs.

The Rational and the Superrational, Vol. II. By C. J. Keyser. New York, Scripta Mathematica. 8+259 pages. $4.25.

A Course of Pure Mathematics. Tenth Edition (with index). By G. H. Hardy. New York, Cambridge University Press. 12+509 pages. $4.75.

Sur Les Cercles Bitangents a la Parabole. By A. Durand. Paris, France, Librairie Vuibert, 1952. 32 pages.

Tables of Coulomb Wave Functions, Vol. 1, Applied Mathematics Series 17. National Bureau of Standards. 141 pages. $2.00.

Tables for the Analysis of Beta Spectra, Applied Mathematics Series 13. National Bureau of Standards, 1952. 61 pages. 35 cents.

Advanced Statistical Methods in Biometric Research. By C. R. Rao. New York, John Wiley and Sons Co., 1952. xvii+390 pages. $7.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

CLUB NEWS

The Florida State University Mathematics Club was organized on November 6, 1951 with Dr. S. L. Jamison and Dr. H. E. Taylor acting as faculty sponsors.

The Indiana Beta chapter of Kappa Mu Epsilon was installed at Butler University on May 16, 1952.

The Louisiana State University chapter of Pi Mu Epsilon presented the fol-

lowing awards on May 1, 1952: senior award, Jasper A. Welch, Jr.; freshman award, Robert E. Cavanaugh.

The University of Missouri chapter of Pi Mu Epsilon conducted a competitive examination in calculus during 1951–52, awarding the following prizes: James Fithian, $15; Delmar Van Meter, $10; Donald Meyer, $5.

The University of North Carolina chapter of Pi Mu Epsilon has established awards for the two highest scores achieved in a statewide high school mathematics contest. The chapter also has established a library consisting of undergraduate texts and periodicals.

During 1951–52, the *Ricci Mathematical Journal*, official publication of the Ricci Mathematics Academy, Boston College, appeared in November, February, March, and May. Members of the Academy presented the mathematics department with a dissectible cone and a large scale pyramind of plexiglass.

The St. Louis University chapter of Pi Mu Epsilon reports that the sixth Annual Essay Contest for undergraduate students was conducted by Professor Lorenz. The senior division was won by Mr. Ying-nien Yu, a student at Parks College. His paper was entitled, *Daniel Bernoulli's hydrodynamical equation.* Mr. Roland Nokes also of Parks College won the junior award for his essay, *Daniel Bernoulli.* The Garneau Mathematics Award for the highest ranking senior majoring in mathematics was awarded to Miss Louise Renard.

The mathematics club of the College of Saint Rose, Albany, New York, in April sponsored a High School Geometric Design Contest to engender a keener interest in mathematics among high school students. Over one hundred twenty entries were received from some eighteen high schools in the Albany area.

DO YOU WANT TO BE A MATHEMATICIAN?

Mathematics Department, University of Georgia

1. **Question:** Who should be called a mathematician?
   Answer: The desire among mathematicians themselves is to classify, as mathematicians, only persons who through their own efforts are able to add to existing mathematical theory or aid in the analysis of scientific problems of an advanced nature and contribute materially to their solution.
2. **Question:** What is mathematical research?
   Answer: By research mathematicians mean the creation of new mathematics. More than 6000 pages of mathematical research were published in mathematical journals in the United States during 1949.
3. **Question:** What training should a mathematician have?
   Answer: The mathematician should have advanced post graduate training equivalent to that expected for the Doctor of Philosophy Degree and some additional training in independent research.
4. **Question:** What are the professional advantages of post graduate mathematical training?
   Answer: Professional opportunities for mathematicians may be classified under

(a) university teaching
(b) college teaching
(c) industrial and business employment
(d) governmental employment including employment by the Navy
(e) independent consultation
(f) writing of books

5. **Question:** What does a mathematician earn?
Answer: (a) Salaries paid to mathematicians range from $3000 to $15,000. There are but few mathematicians in the United States whose salary is so large as the latter figure. However, many professions closely allied to mathematics pay even larger salaries. For example, a number of actuaries are paid salaries in excess of $20,000.
(b) Many mathematicians earn considerable sums from royalties on textbooks and other writings. There are several persons in the United States who received as much as $25,000 per year royalties on college textbooks on mathematics.
(c) There is a growing tendency for the United States Navy and many corporations to pay mathematicians for work on special problems. Thus many university professors and others earn considerable sums in addition to their salaries for research work under government or industrial contract.
6. **Question:** Is there a shortage of mathematicians?
Answer: At the present time there is a shortage of mathematicians and it is very difficult to get highly skilled persons to fill positions. At the conclusion of the War a survey made by the Mathematical Association of America showed that demand exceeded supply in the ratio of thirty to one. This situation has considerably improved since that time. However, the demand for mathematicians is still great.
7. **Question:** What are the qualifications necessary for success as a mathematician?
Answer: To succeed as a mathematician one must have a first class analytical mind and willingness to work coupled with a love for the subject.
8. **Question:** How long does it take to become a mathematician?
Answer: The training period for a mathematician is essentially the same as that for any other high class profession, such as medicine. Three or four years of post graduate work in addition to an undergraduate major may be regarded as essential. However, the best men in mathematics, as in any other profession, spend their entire lives on their subject.
9. **Question:** What subjects other than mathematics should a prospective mathematician take in college?
Answer: A person who wishes to become a mathematician should strive first for a good "arts" education. This should include physics, some chemistry, and philosophy. It is necessary for the mathematician to be able to read modern foreign languages and the prospective mathematician should study sufficient French and German to read mathematical literature writ-

ten in these languages. Italian is also desirable and it is likely that in the future Russian will be a great asset.

10. **Question:** What does a mathematician do?
Answer: (a) The mathematician at the University teaches college students and advanced graduate students. He also is continually engaged in his own research or in aiding others with their research.
(b) The mathematician in government employ or in the employ of the Navy or in industry is usually engaged on a research project in conjunction with others, notably engineers and physicists. His contribution to these joint projects is valuable and is as important as that of any other persons working on them.
11. **Question:** How does a mathematician seek employment?
Answer: If a mathematician is an active publisher in his field, employment is apt to seek him. This is particularly true of employment in the university world. Government employment is sought through civil service and industrial employment through contacting the firms in which one is interested.
12. **Question:** What are the opportunities for the mathematician to contribute to the advancement of civilization?
Answer: There is probably no more permanent or useful addition to civilization than mathematical research. The man who substantially adds to mathematical knowledge has the satisfaction that he has added to the advancement of mankind in one of the most useful and enduring ways. Moreover contributions by mathematicians to physics, engineering, astronomy, philosophy, biology, chemistry, and economics advance civilization through advancing these subjects.
13. **Question:** What are some of the satisfactions from mathematics other than salary?
Answer: Satisfactions that come to a mathematician are from his research, from his writing, from his teaching, and from his associations with other people of outstanding intellectual ability.

INTRODUCING CONGRUENCES

R. LARIVIERE, University of Illinois in Chicago

Too many students, taking traditional mathematics courses, complete their mathematical studies before any congruence relations are taught, though the idea of congruence is one of the most rewarding of the simpler mathematical concepts. The introduction of the topic in mathematical clubs and in some of the better classes can remedy this unfortunate situation to some extent.

Some properties of congruences are being used already to explain the familiar criteria of divisibility by the digits and by eleven. Little more is required to provide an explanation not only for these rules but also for lack of a worthwhile rule for divisibility by seven. The basis of the widely used checks of casting out the nines and casting out the elevens in multiplication and division can be included in the discussion of divisibility criteria.

Helping to determine whether the roots of a quadratic equation are rational or not is an especially easy application which is often overlooked. If the discriminant D_1 of an equation is the integer 5322, or even the reduced fraction 5326/933, the last digits show without further computation that the roots of the equation are irrational, for $n^2 \equiv \pm 1,0$ (5). Another equation, which has a discriminant $D_2 = 5339$, also has irrational roots since $(2n+1)^2 \equiv +1$ (8), and the last three digits show that the residue modulo eight is three. Such restrictions are not sufficient, and discriminants like 5329 or 2009 must be subjected to further trial though they satisfy both of the above congruences.

The partial check of equations like the following is another application which can be explained to an eager elementary group. This quick check is obtained by solving the equations modulo some small primes which do not divide the denominator of the values of either of the unknowns.

$$102x + 512y = 317, \quad 231x - 517y = 897; \qquad x = \frac{-623153}{-171006}, \quad y = \frac{6089}{-57002}$$

$$4x + y \equiv 2(7), \quad 0x - (-y) \equiv 1(7); \qquad x \equiv 2(7), \quad y \equiv 1(7).$$

$$2x + 2y \equiv 2(5), \qquad x - 2y \equiv 2(5); \qquad x \equiv 3(5), \quad y \equiv 3(5).$$

$$\frac{623153}{171006} \equiv \frac{6}{3}\,(7) \equiv 2(7) \equiv \frac{3}{1}\,(5) \equiv 3(5).$$

$$\frac{6089}{-57002} \equiv \frac{6}{-1}\,(7) \equiv 1(7) \equiv \frac{-1}{-2}\,(5) \equiv 3(5).$$

Hence the above solutions are satisfactory with respect to divisibility by seven and five.

Editorial Note: This article suggests an important source of club programs. Many exciting and interesting topics, though quite elementary, are postponed to advanced courses which many students do not have an opportunity to elect. Readers are invited to contribute notes on such topics.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

FELLOWSHIPS OF THE EDUCATIONAL TESTING SERVICE

The Educational Testing Service is offering for 1953–54 its sixth series of research fellowships in psychometrics leading to the Ph.D. degree at Princeton

University. Open to men who are acceptable to the Graduate School of the University, the two fellowships each carry a stipend of $2500 a year and are normally renewable.

Fellows will be engaged in part-time research in the general area of psychological measurement at the offices of the Educational Testing Service and will, in addition, carry a normal program of studies in the Graduate School. Competence in mathematics and psychology is a prerequisite for obtaining these fellowships. The closing date for completing applications is January 16, 1953. Information and application blanks may be obtained from: Director of Psychometric Fellowship Program, Educational Testing Service, 20 Nassau Street, Princeton, New Jersey.

PRELIMINARY ACTUARIAL EXAMINATIONS

PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of Part 2 of the 1952 Preliminary Actuarial Examinations are as follows:

First Prize of $200

Freimer, Marshall	Harvard University
Lew, John S.	Yale University

Additional Prizes of $100

Bauman, Norman	Harvard University
Guinane, J. E.	Brown University
Horowitz, Michael	Cornell University
Kerr-Lawson, Angus	University of Toronto
Krause, Ralph M.	Harvard University
Muckenhoupt, Benjamin	Harvard University
Potthoff, Richard	Swarthmore College

Ordinarily only one $200 prize is offered, but this year two candidates tied for first place. Each candidate was awarded a $200 prize.

The Society of Actuaries has authorized a similar set of prizes for the 1953 examinations based on the Part 2 scores.

The Preliminary Actuarial Examinations consist of the following three examinations:

Part 1. *Language Aptitude Examination.*

(Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning.)

Part 2. *General Mathematics Examination.*

(Algebra, trigonometry, coordinate geometry, differential and integral calculus.)

Part 3. *Special Mathematics Examination.*

(Finite differences, probability and statistics.)

The 1953 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service and will be administered by the Society of Actuaries at centers throughout the United States and Canada in May 1953. The closing date for applications is March 15, 1953.

Detailed information concerning the Examinations can be obtained from: The Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

PERSONAL ITEMS

The Honorary Doctor of Science degree was conferred upon Mr. E. C. Molina by the Newark College of Engineering on June 6, 1952.

Albama Polytechnic Institute announces the following: Dr. H. C. Wang of the Institute for Advanced Study has been appointed Research Professor of Mathematics; Dr. W. C. Royster of the University of Kentucky and Dr. J. C. Morelock of the University of Florida have been appointed to assistant professorships; Associate Professor J. C. Eaves has been promoted to a professorship; Assistant Professor V. E. Dietrich has resigned to accept a position as Professor of Physics at North Central College.

Baldwin-Wallace College reports: Assistant Professor J. A. Wilson is now Acting Head of the Physics Department; Assistant Professor Del Willard has accepted a position with the Cadillac Motor Car Company, Cleveland Tank Division.

Georgia Institute of Technology announces: Associate Professors C. W. Hook and I. E. Perlin have been promoted to professorships; Dr. M. H. M. Esser and Dr. G. K. Overholtzer have been appointed to assistant professorships.

Lafayette College makes the following announcements: Assistant Professor C. W. Saalfrank of Rutgers University has been appointed Head of the Department of Mathematics; Mr. Ralph Playfoot and Mr. Samuel Stoddard have been promoted to assistant professorships.

At Marshall College: Assistant Professor Bernice F. Wright has been promoted to an associate professorship; Mr. F. D. White has been appointed to an instructorship.

Yeshiva University announces that The Yeshiva Institute of Mathematics is offering courses leading to the degree of Master of Science in Mathematics. The faculty consists of the following: Professor Herman Baravalle, Adelphi College; R. S. Beard; Professor C. R. Boyer, Brooklyn College; Professor Jesse Douglas, Columbia University; Professor Samuel Eilenberg, Columbia University; Professor A. A. Frankel, Hebrew University, Jerusalem, Israel; Professor H. F. Fehr, Columbia University; Professor Abe Gelbart, Syracuse University; Professor Jekuthiel Ginsburg, Yeshiva University; Professor C. N. Shuster, New

Jersey State Teachers College; and Associate Professor R. C. Yates, United States Military Academy.

Mr. F. S. Badger of Alliance College has been appointed Dean of Students.

Mr. W. H. Badgley, formerly a graduate student at Vanderbilt University, is now a mathematician with Redstone Arsenal.

Mr. D. O. Banks, who has been a graduate assistant at Oregon State College, is a mathematician with Hanford Works, Richland, Washington.

Mr. James Bercos, previously a graduate student at the University of Georgia, has a position as Chief Statistician with the United States Naval Ordnance Plant, Macon, Georgia.

Mr. H. F. Blasch, formerly a student at Hofstra College, is now Research Engineer and Instructor with the United States Army, Fort Monmouth, New Jersey.

Dr. Paul Brock of Reeves Instrument Company has accepted a position as a mathematician with Consolidated Engineering Corporation, Pasadena, California.

Mr. C. C. Buck of the University of Nebraska is now a graduate student at the University of Michigan.

Dean J. W. Clawson of Ursinus College has been named Dean Emeritus and Professor Emeritus of Mathematics.

Associate Professor C. R. DePrima has returned to California Institute of Technology after a year's leave of absence; during the past year he has been at the Office of Naval Research, Washington, D. C.

Dr. A. H. Diamond who has been Visiting Professor of Mathematics at the University of Kansas has been appointed Chief of the Mathematics Branch, Mathematical Sciences Division, Office of Ordnance Research.

Professor G. M. Ewing, who has been on leave of absence from the University of Missouri and has been serving as a mathematician with Sandia Corporation, Albuquerque, New Mexico, has returned to the University.

Dr. J. D. Haggard of Kansas State Teachers College, Pittsburg, has been appointed Visiting Assistant Professor of Mathematics in the College of the University of Chicago for the academic year 1952–53.

Dr. P. C. Hammer, a group leader of hydrodynamics with I. B. M. Corporation, has been appointed to an associate professorship at the University of Wisconsin.

Mr. B. F. Handy is now at the Institute for Numerical Analysis of the National Bureau of Standards, Los Angeles, California.

Dr. Melvin Henriksen of the University of Alabama has been appointed to an instructorship at Purdue University.

Dr. S. H. Herzfeld, who has been doing research for the Office of Rubber Research, Reconstruction Finance Corporation, is now a consultant for the Office of Scientific Research, Air Research and Development Command, Baltimore, Maryland.

Mr. F. A. Kros, formerly a graduate student at the University of Colorado,

has accepted a position as a mathematician with White Sands Proving Ground, Las Cruces, New Mexico.

Professor Walter Leighton is on leave of absence from his position as Chairman of the Department of Mathematics of Washington University and is serving as a consultant for the Office of Scientific Research, Air Research and Development Command, Baltimore, Maryland.

Assistant Professor W. S. Massey of Brown University has been promoted to an associate professorship.

Professor Benjamin Evans Mitchell of Alabama Polytechnic Institute has been appointed to an assistant professorship at Louisiana State University.

Mr. P. S. Null of West Virginia University has accepted a position as Stress Analyst with Douglas Aircraft Company, Long Beach, California.

Dr. Anne F. O'Neill of Smith College has been appointed to an assistant professorship at Wheaton College, Norton, Massachusetts.

Mr. S. T. Paine, formerly of the Department of Meteorology, University of California at Los Angeles, has a position as Research Mathematician with Convair Aircraft Company, San Diego, California.

Dr. D. H. Porter of Indiana University has been appointed to an instructorship at Marion College.

Associate Professor Adrienne Rayl of the Birmingham Center of the University of Alabama has been promoted to a professorship.

Mr. Gordon Ritchie is teaching at Thessalon High School, Ontario, Canada.

Mr. R. W. Royston has been appointed to an assistant professorship at Washington and Lee University.

Associate Professor Robert Schatten of the University of Kansas has been promoted to a professorship.

Dean T. M. Simpson, Jr., of Randolph-Macon College has retired after thirty-three years of service.

Mr. W. B. Stovall, who has been with the Underway Training Element, United States Fleet Sonar School, San Diego, California, is returning to the Bureau of Vital Statistics, State Board of Health, Jacksonville, Florida, in the position of Chief.

Mr. R. E. Thomas of State University of Iowa has accepted a position as a mathematician with Battelle Memorial Institute, Columbus, Ohio.

Mr. M. W. Tolman, previously a student at the University of South Dakota, has a position as Junior Geophysicist with Shell Oil Company, Tulsa, Oklahoma.

Assistant Professor J. W. Warwick of Wellesley College has accepted a position as Scientific Consultant with the Harvard College Observatory, High Rolls, New Mexico.

Dr. J. G. Wendel of the Rand Corporation has been appointed to an assistant professorship at Louisiana State University.

Reverend B. C. Zimmerman who has been a Jesuit Missionary of Cayo, British Honduras, Central America died on April 22, 1952; he had been a member of the Association for twenty-one years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-THIRD SUMMER MEETING OF THE ASSOCIATION

The thirty-third summer meeting of the Mathematical Association of America was held at Michigan State College, East Lansing, Michigan, on Monday and Tuesday, September 1–2, 1952, in conjunction with the summer meetings of the American Mathematical Society, the Institute of Mathematical Statistics, the Econometric Society, and the Pi Mu Epsilon Fraternity. A total of seven hundred and thirty-four adults were registered, including the following three hundred and twenty-one members of the Association.

C. R. Adams, M. W. Al-Dhahir, Bess E. Allen, E. B. Allen, W. R. Allen, C. B. Allendoerfer, A. G. Anderson, R. D. Anderson, R. V. Andree, J. J. Andrews, H. A. Antosiewicz, Beulah M. Armstrong, K. J. Arnold, Miriam C. Ayer, W. L. Ayres, H. M. Bacon, J. M. Barbour, I. A. Barnett, C. F. Barr, D. Y. Barrer, R. C. F. Bartels, P. T. Bateman, W. D. Baten, H. M. Beatty, H. W. Becker, E. M. Beesley, E. G. Begle, J. H. Bell, Arthur Bernhart, Dorothy L. Bernstein, T. A. Bickerstaff, R. H. Bing, D. W. Blackett, Shirley A. Blackett, Barbara B. Blair, H. D. Block, C. R. Blyth, R. P. Boas, H. F. Bohnenblust, Julia W. Bower, Angeline J. Brandt, A. T. Brauer, G. U. Brauer, J. L. Brenner, Leonard Bristow, J. C. Brixey, Myrtle C. Brown, R. H. Bruck, C. C. Buck, R. C. Buck, C. E. Buell, R. A. Burrows, J. H. Bushey, Jewell H. Bushey, A. T. Butson, H. E. Campbell, E. L. Canfield, K. H. Carlson, R. E. Carr, W. B. Carver, Jeremiah Certaine, Josephine H. Chanler, Abraham Charnes, Y. W. Chen, S. S. Chern, F. Marion Clarke, Helen E. Clarkson, D. E. Coffee, R. H. Cole, N. B. Conkwright, S. D. Conte, Geraldine A. Coon, T. F. Cope, A. H. Copeland, Max Coral, R. R. Coveyou, V. F. Cowling, H. S. M. Coxeter, J. W. Coy, C. C. Craig, A. B. Cunninghan, Grace M. Cutler, D. A. Darling, Robert Davies, Violet B. Davis, W. M. Davis, R. B. Deal, W. A. Dolid, D. W. Dubois, W. L. Duren, P. S. Dwyer, J. M. Earl, W. F. Eberlein, P. D. Edwards, J. G. Elliott, David Ellis, Benjamin Epstein, D. H. Erkiletian, H. P. Evans, W. H. Fagerstrom, D. T. Finkbeiner, C. D. Firestone, K. W. Folley, J. R. Foote, M. K. Fort, J. S. Frame, Evelyn Frank, C. H. Frick, C. V. Fronabarger, W. B. Fulks, H. K. Fulmer, A. E. Gault, H. M. Gehman, J. J. Gergen, K. G. Getman, J. W. Givens, A. J. Goldman, R. A. Good, A. W. Goodman, Lillian Gough, S. H. Gould, Cornelius Gouwens, Beulah Graham, A. A. Grau, L. M. Graves, R. E. Graves, F. L. Griffin, H. C. Griffith, Betty Grossman, Franklin Haimo, P. R. Halmos, P. C. Hammer, H. W. Handsfield, Frank Harary, M. C. Hartley, E. R. Heineman, R. G. Helsel, M. S. Hendrickson, Anna S. Henriques, I. N. Herstein, Fritz Herzog, G. W. Hess, T. H. Hildebrandt, J. E. Hoffman, R. V. Hogg, D. L. Holl, Aughtum S. Howard, W. J. Huebner, Ralph Hull, J. W. Hurst, W. R. Hutcherson, C. G. Jaeger, R. D. James, T. A. Jeeves, L. W. Johnson, R. E. Johnson, L. S. Johnston, B. W. Jones, P. S. Jones, Wilfred Kaplan, Leo Katz, Dora E. Kearney, M. W. Keller, L. M. Kelly, D. E. Kibbey, E. C. Kiefer, J. M. Kingston, Evelyn K. Kinney, S. C. Kleene, L. A. Knowler, F. W. Kokomoor, W. H. Kruskal, R. M. Lakness, R. E. Langer, Leo Lapidus, E. H. Larguier, H. D. Larsen, Margaret M. LaSalle, C. G. Latimer, J. R. Lee, Walter Leighton, Tadeusz Leser, F. A. Lewis, A. J. Lohwater, C. I. Lubin, R. W. MacDowell, C. C. MacDuffee, Saunders MacLane, H. M. MacNeille, Ella Marth, Beckham Martin, W. T. Martin, Kenneth May, June M. McArtney, Dorothy McCoy, N. H. McCoy, L. E. Mehlenbacher, B. E. Meserve, D. M. Mesner, E. J. Mickle, W. E. Milne, Benjamin Ernest Mitchell, Josephine M. Mitchell, Harriet F. Montague, Mabel D. Montgomery, D. C. Morrow, F. C. Mosteller, H. T. Muhly, C. W. Munshower, W. L. Murdock, S. B. Myers, Zeev Nehari, A. L. Nelson, Ivan Niven, M. L. Norden, E. A. Nordhaus, E. S. Northam, Ilse L. Novak, D. G. O'Connor, Ruth E. O'Donnell, E. G. Olds, R. R. Otter, J. C. Oxtoby, C. D. Parker, Mary H. Payne, L. L. Pennisi, F. W. Perkins, H. P. Pettit, George Piranian, A. E. Pitcher, J. C. Polley, J. E. Powell, G. B. Price, C. B.

Rader, Tibor Rado, Henry Rainbow, J. F. Randolph, G. E. Raynor, Mina S. Rees, P. V. Reichelderfer, W. T. Reid, C. E. Rhodes, H. B. Ribeiro, F. A. Rickey, P. R. Rider, D.D. Rippe, J. H. Roberts, R. A. Rosenbaum, P. C. Rosenbloom, Arthur Rosenthal, M. F. Rosskopf, E. H. Rothe, Charles Saltzer, Hans Samelson, A. C. Schaeffer, R. D. Schafer, J. A. Schatz, Henry Scheffe, E. D. Schell, Edith R. Schneckenburger, Augusta L. Schurrer, C. H. W. Sedgewick, Esther Seiden, D. H. Shaffer, M. E. Shanks, Sister Mary Henrietta, R. L. Shively, Edward Silverman, M. F. Smiley, G. W. Smith, Ernst Snapper, W. S. Snyder, Elizabeth S. Sokolnikoff, C. E. Springer, E. P. Starke, F. H. Steen, H. E. Stelson, C. F. Stephens, B. M. Stewart, Ruth W. Stokes, R. R. Stoll, E. H. Stone, M. H. Stone, H. C. Stotz, D. D. Strebe, P. C. Sweetland, Gabor Szego, A. D. Talkington, O. E. Taulbee, Mildred E. Taylor, William Clare Taylor, H. P. Thielman, W. J. Thron, H. E. Tinnappel, John Todd, Leonard Tornheim, A. W. Tucker, Bryant Tuckerman, J. W. Tukey, Eugene Usdin, E. P. Vance, G. W. Walker, R. M. Walter, S. S. Walters, D. L. Webb, G. C. Webber, C. P. Wells, Irene L. Wente, W. M. Whyburn, L. R. Wilcox, J. E. Wilkins, G. P. Williams, Y. K. Wong, A. W. Wortham, R. L. Young, B. K. Youse, J. H. Zant, W. M. Zaring, R. K. Zeigler, J. L. Zemmer, J. A. Zilber.

Sessions of the Association were held on Monday afternoon at 2:00 with President Saunders MacLane presiding, on Tuesday morning at 9:00 with Vice-President Jewell H. Bushey presiding, and on Tuesday evening at 7:30 with Vice-President F. L. Griffin presiding. All sessions were held in Room 118, Physics-Mathematics Building of Michigan State College. A feature of the sessions was the first series of Earle Raymond Hedrick Lectures delivered by Professor Tibor Rado. The Program Committee for the meeting consisted of E. P. Vance, Chairman, O. E. Lancaster, and Kenneth May.

FIRST SESSION OF THE ASSOCIATION

"Illustrations of Game Theory," by Professor Harold W. Kuhn, Princeton University.

"On Minimizing the Oscillation of a Function," by Dr. J. E. Wilkins, Nuclear Development Associates. (Presented by title.)

"What Machines Cannot Do," by Professor P. C. Rosenbloom, University of Minnesota.

The Earle Raymond Hedrick Lectures: "Derivatives and Jacobians," Lecture I, by Professor Tibor Rado, Ohio State University.

SECOND SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Derivatives and Jacobians," Lecture II, by Professor Tibor Rado, Ohio State University.

Symposium on the Mathematical Training of Engineers, Chairman, Professor Ralph Hull, Purdue University.

"Mathematics and the Engineering Curriculum," by Professor E. B. Allen, Rensselaer Polytechnic Institute.

"Mathematics Used for Engineering Designs," by Dr. C. H. Harry, Bureau of Aeronautics, Department of the Navy.

"Problems of Mathematicians Who Teach Engineers," by Professor Elizabeth S. Sokolnikoff, University of Wisconsin.

THIRD SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Derivatives and Jacobians," Lecture III, by Professor Tibor Rado, Ohio State University.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the Mathematics Library in the Physics-Mathematics Building. This meeting was preceded by joint meetings of the Executive and Finance Committees on Sunday evening and Monday morning. Of the forty members of the Board, there were thirty present, including all the officers of the Association. Among the more important items of business transacted were the following:

Professor W. H. Fagerstrom was present to discuss the contests for high school students which have been conducted for several years by a committee of the Metropolitan New York Section. The Board voted to commend the section on its activity in this field and to recommend that it continue to conduct contests on approximately the present scale, taking care to avoid conflict with analogous activities of other sections and other organizations.

The Board voted to authorize the publication of a joint List of Members with the American Mathematical Society. It is expected that copies will be distributed to members of the Association in odd-numbered years.

On the recommendation of the Executive Committee, the Board voted to re-elect Professor Edith R. Schneckenburger of the University of Buffalo as Associate Secretary of the Association for a second term of five years, beginning on January 1, 1953.

The Board of Governors voted to instruct the Secretary to send copies of the following resolution to the Governor of the State of Oklahoma, to the presiding officers of the two houses of the Oklahoma Legislature, to the Attorney General of Oklahoma, and to the Presidents of Oklahoma A and M College and the University of Oklahoma, and to read the resolution at the next session of the Association and to publish the resolution in full in the account of this meeting:

"The Board of Governors of the Mathematical Association of America have considered the effect upon mathematics and mathematicians of the Oklahoma Loyalty Oath Law of April 9, 1951.

"The Board wishes to convey to the heads of government in Oklahoma their view that many mathematicians regard this type of legislation as dangerous to individual liberty and to the principles which protect freedom of thought in a university from political infringement, and as ineffective for the purpose of eliminating disloyal influences from the universities. Even those who sign such oaths often do so more in a spirit of tolerance for the patriotic motives which conceived the law than in respect for the law itself, or more in fear than in concurrence.

"Whenever such laws are enacted, it is to be expected that at least a few of the American idealists, the sincerely religious people, and the courageous lovers

of liberty will rebel, with tragic effects upon themselves, their families, and upon the universities where they work. It is not to be expected that such legislation will be effective in eliminating from the faculties men who are dangerous to the national welfare, so that the injury caused is a useless waste, as at Oklahoma A and M College, where a department of mathematics which had achieved much recognition for its mathematical work was seriously damaged."

MEETINGS OF OTHER ORGANIZATIONS

The sessions of the American Mathematical Society began on the afternoon of Tuesday, September 2, and continued through Friday. The Colloquium lectures were delivered by Professor Alfred Tarski of the University of California on the topic: "Arithmetical classes and types of algebraic systems." An invited address with the title "On the topology of 3-manifolds" was delivered by Professor E. E. Moise of the University of Michigan.

The Institute of Mathematical Statistics and the Econometric Society held sessions beginning on Tuesday morning and continuing through Friday afternoon. The Pi Mu Epsilon Fraternity held its sessions on Sunday evening and on Monday morning and evening.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of J. S. Frame, Chairman; J. H. Bell, H. M. Gehman, Fritz Herzog, Leo Katz, E. A. Nordhaus, H. E. Stelson, B. M. Stewart, J. W. T. Youngs.

Registration headquarters were in the lobby of Phillips Hall, and accommodations for arrivals were available at all hours of the day or night. Rooms in Phillips Hall and Snyder Hall were open for occupancy from Sunday afternoon, August 31, until Saturday noon, September 6. Meals were served in the cafeteria of Phillips Hall.

An informal coffee hour was held on Monday evening in the Conference Room of the Physics-Mathematics Building. There was an informal tea on Tuesday afternoon in Phillips Lower Lounge. Both of these events were arranged by the members of the Michigan State College Mathematics Department and their wives.

The banquet for the mathematical organizations was held on Wednesday evening in the Union building. Professor C. B. Allendoerfer acted as toastmaster, and introduced representatives of each of the five mathematical organizations. President John A. Hannah of Michigan State College spoke on the scientific aspects of the Point Four Program. Professor J. J. Gergen of Duke University presented a resolution of thanks to our hosts for their warm hospitality and excellent arrangements, which was adopted by an enthusiastic vote.

On Thursday evening members of the Department of Music of Michigan State College presented a Sonata Recital including classical and modern music for the bassoon, oboe, viola, violin and piano.

The annual party of the Institute of Mathematical Statistics took place on Thursday evening at the Coral Gables Ballroom.

Conducted tours of the campus were held on Tuesday and Wednesday mornings, and there was a trip through the Oldsmobile plant on Wednesday afternoon. On Thursday was held an all day trip to the Henry Ford Museum and Greenfield Village in Dearborn.

An exhibit of mathematical textbooks was available during the week in the Physics-Mathematics Building.

All those who attended the meeting are grateful to the members of the staff of Michigan State College for their hospitality and for the efficiency with which the details of the meeting were carried out.

H. M. GEHMAN, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Annual Meeting, Washington University, St. Louis, Missouri, December 30, 1952.

Thirty-fourth Summer Meeting, Laval University, Quebec, Canada, August 31–September 1, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May, 1953.

ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.

INDIANA, Ball State Teachers College, Muncie, May 2, 1953.

IOWA, Cornell College, Mount Vernon, April 17–18, 1953.

KANSAS

KENTUCKY, University of Louisville, Spring, 1953.

LOUISIANA-MISSISSIPPI, Millsaps College, Jackson, Mississippi, February 13–14, 1953.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Howard University, Washington, D. C., December 6, 1952.

METROPOLITAN NEW YORK, Spring, 1953.

MICHIGAN, Western Michigan College of Education, Kalamazoo, April 11, 1953.

MINNESOTA

MISSOURI, William Jewell College, Liberty, Spring, 1953.

NEBRASKA

NORTHERN CALIFORNIA

OHIO

OKLAHOMA

PACIFIC NORTHWEST, Montana State University, Missoula, June, 1953.

PHILADELPHIA, University of Delaware, Newark, November 29, 1952.

ROCKY MOUNTAIN, University of Colorado, Boulder, April, 1953.

SOUTHEASTERN, Alabama Polytechnic Institute, Auburn, March 13–14, 1953.

SOUTHERN CALIFORNIA, Los Angeles City College, March 14, 1953.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanic Arts, State College, January 2–3, 1953.

TEXAS

UPPER NEW YORK STATE, United States Military Academy, West Point, Spring, 1953.

WISCONSIN, Mount Mary College, Milwaukee, May, 1953.

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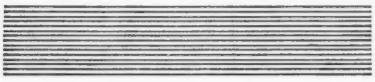

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